## A NEW DEFERMENT PROCEDURE FOR MATHEMATICIANS

In April, 1943, a deferment procedure for mathematicians, involving the National Committee on Physicists and Mathematicians, was authorized under Activity and Occupation Bulletin No. 35. This was explained fully in a memorandum distributed to chairmen of departments of mathematics on May 11, 1943, and reproduced in the July issue of this BULLETIN and in the June-July issue of the *American Mathematical Monthly*. By this procedure employers were advised to send the original copy of form 42A to the National Committee for evaluation. From the Committee, the form was returned to the local board with an appropriate statement by the Committee concerning the advisability of the deferment. The National Committee was also empowered to appeal cases of registrants for whom occupational deferment was not granted by the local board. Under this arrangement, the Committee has performed its duties well and effectively.

Activity and Occupation Bulletin No. 35, which authorizes the activities of the National Committee, is to be rescinded in the near future, over the strong protests of representatives of mathematics and physics. The Committee will therefore cease to exist and will no longer be available to advise local boards on problems connected with the deferment of mathematicians. The new procedure for persons engaged in the occupations defined as critical in Local Board Memorandum No. 115, revised August 16, 1943, is described in Local Board Memorandum 115B which is reproduced at the end of this memorandum. *Only those cases in which occupational deferment is refused by the local board will become involved in this procedure.*

In the August 16 revision of Local Board Memorandum 115, a list of critical occupations, which includes mathematicians, was set up.

The following represents a summary of the occupational deferment procedure for the critical occupations:

1. The employer continues, as in the past, to present his case for deferment to the local board through the use of Form 42A.

2. The local board will arrive at a classification for the registrant on the basis of the evidence submitted to it by the employer. It may, if it so desires, consult with the local United States Employment Service Office concerning classification. (LBM 115 and LBM 115-C)

3. The local board will then make its decision. If occupational deferment is granted, no further action is involved during the period covered by the deferment. If, however, deferment is refused, and the registrant is placed in I-A, the case is continued as follows.

4. *If no appeal is taken* within the 10-day period allowed for this purpose, the local board is *directed* to refer the case to the local (local with respect to the registrant's local board) office of the United States Employment Service and a 30-day stay of induction is granted to permit action by this office.

5. *If an appeal is taken* and the local board's decision is reversed, the registrant will be reclassified, and the case is closed for the duration of the deferment period. If, however, the I-A classification made by the local board is supported by the appeal board, the case *must be* referred to the local U.S.E.S. office.

6. When the local U.S.E.S. office receives a case from the local board either directly (4, above), or after appeal (5, above), it may certify to the local board either that

a. the registrant should be deferred in his present position, or

b. that they have succeeded in placing the registrant in a new position.

In the first case the local board is directed to reopen the classification of the registrant, and presumably will grant occupational deferment. In the second case, a further period of 10 days is given the registrant for the filing of a new Form 42A requesting occupational deferment in his new position, and on the basis of which the local board shall reopen the classification.

7. If the U.S.E.S. fails to present any notification to the local board during the period allowed, or if it certifies that the registrant is needed in new employment, but does not succeed in placing the man in such employment, the local board may proceed with the induction of the registrant.

8. The local offices of the U.S.E.S. can refer cases reaching them and involving persons in "critical" professional and scientific occupations to the National Roster of Scientific and Specialized Personnel. The details of the relation between the U.S.E.S. offices and the Roster have not been fully defined.

The following suggestions are made to facilitate the proper functioning of the procedure herein outlined.

(1) It becomes of even greater importance than heretofore that an employer's original presentation of a case to the local board for occupational deferment or continuation of deferment be made as strong as possible. No opportunity for strengthening a case should be overlooked.

(2) The employer having once requested occupational deferment should be prepared to appeal every case in which such deferment is refused by the local board. This again implies that the original presen-

tation should be as strong as possible, so as to stand up under an appeal.

(3) If an appeal does not secure deferment, the employer should take it upon himself to see that the case is referred to the local U.S.E.S. office, as directed by Local Board Memorandum 115-B. It is possible that the local board may not be familiar with the requirement that they do this.

(4) The employer should take steps to insure that a case which has reached the U.S.E.S. office is actually referred by it to the National Roster.

(5) It should be noted that there is no clear statement as to whether an appeal may be taken *after* the action of the U.S.E.S. Presumably an appeal could be taken at this time, but it is implied in LBM 115-B that the proper time for an appeal is immediately (within 10 days) after the local board's original I-A classification.

(6) In case the procedures here listed have been followed and misclassification of important personnel nevertheless results, the National Roster should be informed regarding the particulars of the case.

The Secretary of the Society will be interested in learning of the experiences of department chairmen with the new procedure. Such information will give our representatives in Washington and the War Policy Committee a basis for judging the effectiveness of the plan.

J. R. KLINE,
*Secretary*

October 20, 1943

National Headquarters

SELECTIVE SERVICE SYSTEM

Washington, D. C.

# LOCAL BOARD MEMORANDUM NO. 115-B

ISSUED: 9/1/43

SUBJECT: REGISTRANTS IN CRITICAL OCCUPATIONS

1. *Importance of critical occupations.*—Attached to Local Board Memorandum No. 115, amended August 16, 1943, is a List of Critical Occupations. Only 149 occupations are included on the list, and it is estimated that there are not more than 400,000 registrants of military age throughout the country who have the skills required of the critical occupations. This number is exceedingly small in comparison to the total working force in industry. It can be safely said that the critical occupations as listed by the War Manpower Commission represent the highest levels of skills within industry and that, therefore, all production necessary to the war effort is directly dependent upon the most efficient use within industry of persons qualified in these occupations.

2. *Need for extending every consideration for occupational deferment to registrants in critical occupations.*—It is of the utmost importance that registrants (1) who have the necessary qualifications, (2) who are utilizing them to the fullest extent in a critical occupation in war production or in support of the war effort, and (3) whose removal from their present employment would have an adverse effect upon the maintenance of required production schedules, be given the most serious consideration for extended occupational deferment before being reclassified out of a deferred classification into a class available for service.

3. *Referral to United States Employment Service.*—(a) Local boards are directed that whenever, after the most careful consideration, they conclude a registrant who is alleged to be in a critical occupation in war production or in support of the war effort is not entitled to occupational deferment on the basis of his present employment, to refer the registrant's name and present occupation, together with a statement of his skills and qualifications and place of present employment, to the local office of the United States Employment Service in the area in which the local board is located.

(b) This reference is mandatory and will be accomplished—

(1) If no appeal is taken, upon the expiration of the period afforded the registrant to take an appeal following classification of the registrant in Class I-A, or

(2) If an appeal is taken, upon the return of the file to the local board with the decision that the registrant is continued in Class I-A on appeal.

(c) In all such cases, local boards will delay the issuance of an Order to Report for Induction to such registrant for 30 days from the date of referral to the United States Employment Service.

(d) If, during such 30-day period, the United States Employment Service certifies to the local board that—

(1) The registrant possesses and is fully employing the qualifications required of a critical occupation in war production or in support of the war effort and his removal from his present employment would adversely affect the maintenance of his employer's required production, or

(2) The United States Employment Service has succeeded in placing the registrant in a critical occupation in war production or in support of the war effort with another employer who requires the registrant's skills and qualifications and will fully utilize such skills and qualifications,

the local board will delay the issuance of an Order to Report for Induction to such registrant for a further period of 10 days from the receipt of such certification so as to permit the filing of a new Form 42A requesting the occupational deferment of the registrant.

(e) In the event that either a Form 42A is filed by a new employer with whom the registrant has been placed or certification is received from the United States Employment Service stating that the registrant is necessary in his present employment, the local board will reopen the classification of the registrant and will consider the new evidence as a basis for further occupational deferment.

(f) If, however, no notification is received from the United States Employment Service prior to the expiration of the original 30-day period following referral or, in the event certification is made, that the registrant is needed in new employment but a Form 42A is not received before the expiration of the 10-day period allowed for the filing of a new claim for deferment, the local board may proceed with the induction of the registrant.

4. *Cooperation by United States Employment Service.*—The War Manpower Commission is informing the United States Employment Service of the need for expediting investigation of the cases of any registrants qualified for critical occupations which are referred to it by local boards. Even though a registrant is not employed in the area in which his local board is located, reference by the local board will, nevertheless, be to the local United States Employment Service office in the local board area, and any further reference of the case to the United States Employment Service office in the area in which the registrant is employed will be undertaken and accomplished by the United States Employment Service.

5. *Registrants between ages of 18 and 25 in critical occupations.*—Men who are utilizing their full skills in critical occupations when the necessity for their continued retention is clearly demonstrated by their employer should be considered as registrants who are utilizing the highest skills or professional qualifications, which is the exception to the general policy expressed in Local Board Memorandum No. 158-A, requiring the release of registrants between the ages of 18 and 25 during the first 6 months of a replacement schedule. In the lower age brackets, however, employers should be required to supply complete evidence that the registrant's qualifications and skills are exceptional, are being fully utilized, and cannot be replaced.

LEWIS B. HERSHEY,
*Director*

## DEFERMENT OF A.S.T.P. TEACHERS

The following is a copy of an undated notice sent through Service Commands to presidents of institutions participating in or scheduled to participate in the Army Specialized Training Program. It is not known as yet whether the instructions given therein are modified or altered by the provisions of LBM 115-B.

### NOTICE CONCERNING DEFERMENT OF ASTP TEACHERS

(Sent through Service Commands)

To Presidents of Institutions Participating in or Scheduled to Participate in the Army Specialized Training Program:

It is of the utmost importance that qualified and irreplaceable teachers who are at institutions under contract to the Army Specialized Training Program and who are providing instruction under the program be permitted to continue this work.

The requirements of the Army Specialized Training Program indicate that very serious disruption is probable if the responsible officials of these institutions do not sufficiently emphasize to Selective Service Authorities the necessity for the occupational deferment of these teachers.

As a first step, it is requested that all teachers in institutions under contract to the Army Specialized Training Program register their qualifications with the National Roster of Scientific and Specialized Personnel, Washington, D. C. The National Roster is concerned with the proper utilization of professional personnel and is an important source of information in this matter to the Selective Service System.

The Selective Service Form 42-A, completely filled out, should be used for individual teachers for whom deferment is considered necessary, to furnish the appropriate local board with specific information reflecting the considered judgment of the institution with respect to qualifications, degree of training, and experience required to engage in the profession; and to inform the board either (1) that no replacement can be obtained and that replacement training time is too great to afford relief, or (2) that no replacement can be obtained immediately, in which case the time required for replacement should be indicated. Special attention should be devoted to furnishing complete information concerning the capacities and duties of the individual teacher. In cases where teachers can be replaced, it should be emphasized that the replacement should not break into the term during which the incumbent teacher is giving instruction.

It is essential that all possible means of procuring replacement be employed, including the transfer of non-vulnerable teachers who are engaged in civilian training, the conversion of teachers from cognate fields or from administrative duties, the recruitment of teachers not now engaged in essential activities, and in general the maximum practical use of existing teaching personnel.

Where deferment is not obtained at the local board level, appeal should be made within ten days to the area or district appeals board and if necessary the case should be called to the attention of the State Director with a request for a Presidential appeal.

You are requested to adhere strictly to all Selective Service procedures regarding occupational deferment. You are also requested to advise the Military Commandant of your institution immediately when any irreplaceable teacher is classified by Selective Service authorities as available for military service.

In the event that the Military Commandant at the institution considers the deferment of the teacher to be essential to the success of the Army Specialized Training Program, representations to that effect may be made by him to the State Director of the Selective Service System. Such representations are not to be made, however, unless all appeal procedure provided under the Selective Service Act is being pursued and unless in fact the particular teacher is irreplaceable in the institution.

In the event that the deferment is denied after all appeal steps through the state level have been complied with, the institution should immediately transmit the matter to the Labor Branch, Industrial Personnel Division, Headquarters, Army Service Forces, for representations at the Washington level.

In the case of the irreplaceable teacher whose induction into the Army has taken place, causing serious jeopardy to the Army Specialized Training Program, institutions may submit a full statement of the facts to the Labor Branch, Industrial Personnel Division, Headquarters, Army Service Forces, Washington, D. C., for consideration of methods for remedial action.

## COMMITTEE ON AVAILABLE TEACHERS OF COLLEGIATE MATHEMATICS

The Committee on Available Teachers of Collegiate Mathematics, established by the War Policy Committee of the American Mathematical Society and the Mathematical Association of America, has been in existence since the beginning of April, 1943. During this time it has received and answered numerous inquiries from colleges and universities needing teachers of mathematics, as well as from teachers who were free to accept appointments.

A total of 149 persons have registered with the Committee. Of this number, 129 have at least the master's degree. The Committee has answered requests for teachers of mathematics from 67 different institutions. To these institutions a total of 348 names, representing 111 different persons, have been suggested. According to the Committee's records, 57 of the persons registered are no longer available. It should be pointed out that of the remaining candidates some have indicated that they are available only for summer or part-time teaching, others have had only secondary school teaching experience, while others are available for appointment in restricted geographical areas.

It is anticipated that the demand for teachers will increase considerably during the next two or three months. On the other hand, the number of available well-qualified candidates who have registered with the Committee for such appointments has been reduced to such an extent that the remaining supply has become quite inadequate to meet the expected demand.

For this reason, the committee requests that (a) individual teachers report their availability and (b) departments of mathematics inform the Committee at the earliest possible date of their needs during the next half year, giving as full details as they can concerning the qualifications expected, the salary offered and other pertinent facts relating to their vacancies, and of members who are free to fill temporary positions in other institutions.

The effectiveness of the work of this Committee will be enhanced if registrants will report positions which they have accepted and if inquiring institutions will report appointments that have been made.

Committee on Available Teachers,
110 Bennett Hall, University of Pennsylvania
W. D. CAIRNS
ARNOLD DRESDEN
J. R. KLINE

September 2, 1943.

# COMPLEX METHODS IN THE THEORY OF FOURIER SERIES

A. ZYGMUND

1. **General remarks.** Two new ideas which greatly influenced the theory of Fourier series in this century are the Lebesgue integral and the applications of complex functions. The original impetus due to the discoveries of Lebesgue would have been spent long ago, but for the fact that its combination with complex methods opened entirely new prospects for trigonometric series.

The essential tool of Lebesgue theory is the fact that the integral is differentiable almost everywhere and that the derivative is equal to the integrand almost always. Most of the fundamental results of the theory of trigonometric series which were based on that fact had been known, roughly, before 1920. Although some important results have been discovered since then, the progress of purely real methods in the last twenty odd years has been relatively slow and limited to isolated problems. It seems quite likely that the structure of real functions must be investigated in more detail before purely real methods can resume their progress. On the other hand, it seems that the complex variable approach to many problems of the theory is the most natural one and may even be of considerable help in the analysis of the structure of real functions.

Every trigonometric series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \tag{1}$$

is the real part of the power series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu - ib_\nu)z^\nu \tag{2}$$

on the unit circle $z=e^{i\theta}$. The imaginary part of the series (2) for $z=e^{i\theta}$ is the series

$$\sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta) \tag{3}$$

and is called the *conjugate* of (1).

Similarly, the harmonic function

An address delivered before the New Brunswick meeting of the Society on September 12, 1943; received by the editors September 12, 1943.

$$u(r,\theta)=\tfrac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu\theta+b_\nu\sin\nu\theta)r^\nu \tag{4}$$

associated with the series (1) is the real part of the analytic function $\phi(z)$, $z=re^{i\theta}$, defined by the series (2). The harmonic function

$$v(r,\theta)=\sum_{\nu=1}^{\infty}(a_\nu\sin\nu\theta-b_\nu\cos\nu\theta)r^\nu \tag{5}$$

associated with the series (3) is conjugate to the function $u(r,\theta)$ and is the imaginary part of the function $\phi(z)$.

Thus the problems of trigonometric series may be treated as problems (boundary value problems) of the theory of analytic functions. By complex methods in the theory of trigonometric series we however mean something more special, namely the application of the methods of analytic functions and in particular of the fact that the latter form a field. Elementary operations performed on analytic functions lead to analytic functions, as does also the operation of taking a function of a function. Nothing like that holds for harmonic functions, since even the square of a harmonic function need not be harmonic. Thus dealing directly with analytic functions instead of with their real parts gives obvious advantage.

The complex methods in trigonometric series have been systematically developed in the last quarter century, although some isolated applications can be traced back to an earlier period. Roughly speaking, in the development of complex methods we may discern three major trends:

(a) The method of the classes $H^p$,
(b) The method of conformal representation,
(c) The Littlewood-Paley method,

and it is the purpose of this talk to say a few words about each of these methods. It goes without saying that in a talk like this the presentation may be only very sketchy and must be limited to a discussion of a few particular results.

2. **Classes $H^p$ of analytic functions.** One of the important problems of the theory of trigonometric series is to establish conditions under which a given trigonometric series (1) is a Fourier series. In other words: when is there an integrable function $f(x)$ such that the coefficients $a_n$, $b_n$ are given by the familiar formulas

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx,\qquad b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx? \tag{6}$$

In the theory of Fourier series we often consider besides the most general integrable functions (by "integrable" we always mean $L$-integrable) classes of more special functions, for example, continuous, bounded, of the Lebesgue class $L^p$, $p \geqq 1$, and so on, and we may ask, in addition, under what condition does $f$ belong to one of those classes. The Fourier character of the series (1) may be easily detected by means of the harmonic function (4) associated with the series. For functions of the class $L^p$, $p>1$, we have a very simple test: a necessary and sufficient condition that (1) is the Fourier series of a function $f$ of the class $L^p$ is that the integral

$$\int_0^{2\pi} |u(r, \theta)|^p d\theta \tag{7}$$

be bounded for $0 \leqq r < 1$. For $p=1$ this result is no longer true: a necessary and sufficient condition for the boundedness of the integral

$$\int_0^{2\pi} |u(r, \theta)| d\theta \tag{8}$$

is that there exist a function $F(x)$, $0 \leqq x \leqq 2\pi$, of bounded variation and such that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos nx dF(x), \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \sin nx dF(x). \tag{9}$$

Of course, if $F(x)$ is absolutely continuous and $F'(x)=f(x)$, these formulas reduce to (6), but in general it is not true. The series (1) with coefficients (9) is called a *Fourier-Stieltjes series*. The series (1) is a Fourier-Stieltjes series if and only if the integral (8) is bounded for $0 \leqq r < 1$. A necessary and sufficient condition that (1) should be an ordinary Fourier series is slightly more complicated: it is

$$\lim_{r, r' \to 1} \int_0^{2\pi} |u(r, \theta) - u(r', \theta)| d\theta = 0. \tag{10}$$

(For the proofs of all these results see, for example, Evans [2][1] or Zygmund [33]; in the sequel, the latter book will be quoted TS.)

Let us now consider any function

$$\phi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu \tag{11}$$

[1] Numbers in brackets refer to the references listed at the end of the paper.

regular for $|z|<1$. If the integral

$$\int_0^{2\pi} |\phi(re^{i\theta})|^p d\theta \tag{12}$$

(analogous to (7)) is bounded for $0 \leqq r < 1$, the function $\phi(z)$ is said to belong to the class $H^p$ ($H$ stands for Hardy); $p$ is here any positive number. The case $p=2$ is of special interest since in this case the integral (12) is easily expressible in terms of the coefficients $c_\nu$ by means of the Parseval formula

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^2 d\theta = \sum_{\nu=0}^{\infty} |c_\nu|^2 r^{2\nu}.$$

Hence the function $\phi(z)$ belongs to the class $H^2$ if and only if the series $\sum |c_\nu|^2$ converges. No result of this kind holds for $p \neq 2$, and this makes $H^2$ a central class ("central" in more than one sense), whose properties are the easiest to study. The fact that if $\sum |c_\nu|^2$ is finite then the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta}$$

is the Fourier series (with respect to the system $\{e^{i\nu\theta}\}$) of a function of the class $L^2$ is the classical Riesz-Fischer theorem.

Of course, given any function $\phi(z)$ regular in $|z|<1$ and of the class $H^p$ we might set

$$\phi^p(z) = \psi^2(z) \tag{13}$$

so that the boundedness of the integral (13) is equivalent to that of

$$\int_0^{2\pi} |\psi(re^{i\theta})|^2 d\theta,$$

that is to say to the fact that $\psi$ is of the class $H^2$, but the formula (13) defines a function $\psi$ regular in $|z|<1$ only if $\phi$ has no zeros there. If $\phi(z)$ does have zeros, the argument has to be modified slightly, and following F. Riesz [26] (TS, [6]) we may proceed as follows. Let $z_1, z_2, \cdots$ ($|z_\nu|<1$) be the zeros (counted according to their multiplicity) of the function $\phi(z) \not\equiv 0$, of the class $H^p$. It may be shown that the product

$$\prod_{\nu=1}^{\infty} |z_\nu| \tag{14}$$

converges. In other words, the sum $\sum(1-|z_\nu|)$ is finite. Conversely, given any sequence, finite or infinite, of numbers $z_\nu$, $|z_\nu|<1$, such that the product (14) converges, there is a function $B(z)$, regular and bounded in $|z|<1$ (and so, in particular belonging to every class $H^p$), having zeros at the points $z_\nu$ and only there. If, for example, the points $z_\nu$ are all different from the origin, the function $B(z)$ may be defined as the product ("Blaschke product")

$$B(z) = \prod_\nu \frac{z - z_\nu}{z - z_\nu^*} \cdot \frac{1}{|z_\nu|} \tag{15}$$

where $z_\nu^* = 1/\bar{z}_\nu$, is the point conjugate to $z_\nu$ with respect to the circumference $|z|=1$. (If $\phi$ has a zero of order $k$ at the origin, we have to insert the factor $z^k$ on the right of (15).) It may be easily shown that $|B(z)|<1$ for $|z|<1$. Thus the function $\phi(z)/B(z)=\psi(z)$ is regular for $|z|<1$, does not vanish there, and we have the decomposition

$$\phi(z) = B(z)\psi(z).$$

If $\phi(z)$ has no zeros we set $B(z)\equiv 1$. It is an important fact that the function $\psi$ also belongs to $H^p$ (more precisely, if for the function $\phi$ the integral (12) does not exceed a constant $M$ for all $r<1$, the function $\psi$ has the same property). Since we may write

$$\phi(z) = \psi(z) + (B(z) - 1)\psi(z) = \psi(z) + \psi_1(z)$$

say, and since $B(z)-1$ does not vanish in $|z|<1$ and is absolutely less than 2 there, we get the following decomposition theorem: *every function of the class $H^p$ may be represented as a sum of two functions of the class $H^p$ which have no zeros in $|z|<1$.* Since the functions of the class $H^p$ and without zeros are reducible to functions of the class $H^2$, whose properties are particularly simple, those properties may be extended to classes $H^p$. In particular, we get the following fundamental theorem (in which by a non-tangential path we mean any continuous curve approaching a point $z_0$, $|z_0|=1$, from inside the unit circle and contained between two chords through $z_0$ of that circle).

*Suppose that $\phi(z)$ is of the class $H^p$. Then for almost every point $e^{i\theta}$ on $|z|=1$,*

$$\lim_{z\to e^{i\theta}} \phi(z) \tag{16}$$

*exists and is finite provided that $z$ approaches $e^{i\theta}$ along any non-tangential curve. Moreover, if $\phi(e^{i\theta})$ denotes the limit* (16),

$$\int_0^{2\pi} |\phi(re^{i\theta}) - \phi(e^{i\theta})|^p d\theta \to 0, \tag{17}$$

$$\int_0^{2\pi} |\phi(re^{i\theta}) - \phi(r'e^{i\theta})|^p d\theta \to 0, \tag{18}$$

*as $r$ and $r'$ tend to* 1.

It must be added that for $p>1$ this result had been known before the decomposition theorem was proved (for then, by what was said before, the series $\sum c_\nu e^{i\nu\theta}$ is a Fourier series, and so may be studied directly by familiar methods), but in the case $0<p\leqq 1$ it brings to light some new facts.

The most important case here is that of $p=1$. Suppose that $\phi(z)$ belongs to $H$ (that is $H^1$), say

$$\int_0^{2\pi} |\phi(re^{i\theta})|\, d\theta < M \tag{19}$$

for $0\leqq r<1$. The function $\phi(z)$ is a (complex-valued) harmonic function. Thus (by what was said before) the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta} = \sum_{\nu=0}^{\infty} c_\nu(\cos\nu\theta + i\sin\nu\theta) \tag{20}$$

is a Fourier-Stieltjes series. On the other hand, since (19) implies (18) with $p=1$, we see that the series (20) is an ordinary Fourier series. Thus it turns out that for the trigonometric series (20) which are generated by power series on the circle of convergence, the distinction between ordinary Fourier series and Fourier-Stieltjes series disappears. If we take into account the familiar fact [TS, 16] that Fourier-Stieltjes series are obtained by differentiating formally Fourier series of functions of bounded variation we obtain the following result.

*If the series*

$$\sum_{\nu=0}^{\infty} C_\nu e^{i\nu\theta} \tag{21}$$

*is the Fourier series of a function of bounded variation, this function must be absolutely continuous. In particular, if $\Phi(z)$ is regular in $|z|<1$, continuous in $|z|\leqq 1$, and if $\Phi(e^{i\theta})$ is of bounded variation, then $\Phi(e^{i\theta})$ is absolutely continuous* (F. and M. Riesz [27]).

This fact is of great importance for the theory of conformal mapping. Another result which may be obtained by means of the decomposi-

tion theorem, and which shows the difference that exists between ordinary trigonometric series and the series (20), is the following theorem.

*If* (21) *is the Fourier series of a function of bounded variation the series* $\sum|C_\nu|$ *converges* (Hardy and Littlewood [5]).

There exist such simple proofs of this result that one of them may be reproduced here (See Hardy, Littlewood and Pólya [7]). The problem reduces to showing that if the function $\phi(z)=\sum c_\nu z^\nu$ is of the class $H$, the series $\sum|c_\nu|/(\nu+1)$ converges. We may assume that $\phi(z)$ has no zeros in $|z|<1$, so that $\phi=\psi^2$ where $\psi(z)=\sum d_\nu z^\nu$ is of the class $H^2$. If we use the fact that $\sum|d_\nu|^2<+\infty$, the convergence of $\sum|c_\nu|(\nu+1)^{-1}$ follows from the fact that

$$\sum_{\nu=0}^{\infty}(\nu+1)^{-1}|c_\nu| = \sum_{\nu=0}^{\infty}(\nu+1)^{-1}\left|\sum_{k+l=\nu} d_k d_l\right|$$

$$\leqq \sum_{\nu=0}^{\infty}(\nu+1)^{-1}\sum_{k+l=\nu}|d_k d_l|$$

$$= \sum_{k,l=0}^{\infty}|d_k||d_l|/(k+l+1)$$

and from the familiar theorem of Hilbert asserting the convergence of the series $\sum x_k x_l/(k+l+1)$ if $\sum x_k^2$ is finite.

The following theorem due to Hardy and Littlewood [6; TS, p. 249] throws additional light on the behavior of the functions of the class $H^p$.

*Suppose that* $\phi(z)$ *is of the class* $H^p$, *and let*

$$\phi^*(\theta) = \sup_{0\leq r<1}|\phi(re^{i\theta})|.$$

*Then* $\phi^*(\theta)$ *is of the class* $L^p$, *and*

$$\int_0^{2\pi}\{\phi^*(\theta)\}^p d\theta \leqq C_p\int_0^{2\pi}|\phi(e^{i\theta})|^p d\theta,$$

*where* $C_p$ *depends on* $p$ *only.*

Hence every function of the class $H^p$ has a majorant which depends on the argument $\theta$ only and which is of the class $L^p$. An analogous result holds for harmonic functions, and is an important tool in many problems of the theory of Fourier series. A result similar to the theorem just stated holds if instead of $\phi^*(\theta)$ we consider the upper bound of $\phi(z)$ in a sector with vertex at the point $e^{i\theta}$ and directed to the interior of the unit circle.

Let $\phi(z)=u+iv$ be a function regular for $|z|<1$. It is natural to ask for the relation between the behavior of the two integrals

$$\int_0^{2\pi} |u(re^{i\theta})|^p d\theta, \qquad \int_0^{2\pi} |v(re^{i\theta})|^p d\theta.$$

The fundamental theorem of M. Riesz [28; TS, p. 147] asserts that, *if $p>1$, these two integrals are simultaneously bounded or simultaneously unbounded.* Thus, *the series conjugate to the Fourier series of a function of the class $L^p$ is also the Fourier series of a function of the class $L^p$.* The result is not true if $p=1$, that is to say for functions merely integrable.

How the method of the complex variable may be applied to problems of real functions is illustrated by the following example which is by now quite familiar, but deserves mention here. Let $f(\theta)$ be a function of period $2\pi$, and let (3) be the series conjugate to the Fourier series of $f(\theta)$. If $f(\theta)$ is sufficiently regular, for example if it has a continuous derivative, or even if $f(\theta)$ only satisfies a Lipschitz condition of positive order, the series (3) converges to the sum

$$\tilde{f}(\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta+t)\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt = -\frac{1}{\pi}\lim_{\epsilon\to+0}\left(\int_{-\pi}^{-\epsilon}+\int_{\epsilon}^{\pi}\right),$$

which is called the function *conjugate* to $f(\theta)$. Under the conditions imposed on $f$, the integral converges, even absolutely. The problem is whether this integral converges, at least almost everywhere, for the most general function $f$ integrable $L$? That this is so for $f$ continuous was shown already by Fatou [3] and even that rather special result is far from obvious. For the most general integrable $f$ it is easy to show that almost everywhere the existence of $\tilde{f}(\theta)$ is equivalent to the existence of the radial limit of the function $\tilde{f}(r, \theta)$ conjugate to the Poisson integral of $f$. If we use rather elementary facts from the theory of the complex variable, we may easily show that the radial limit of $\tilde{f}(r, \theta)$ exists almost everywhere, and this proves that the integral $\tilde{f}(\theta)$ exists almost everywhere (Privaloff [24], Plessner [19]; TS, p. 145).

Although there exist purely real proofs of the existence of $\tilde{f}(\theta)$ (Besicovitch [1], Titchmarsh [30], Marcinkiewicz [14]), the one sketched above is the simplest, and seems to lead to the roots of the matter.

In the case that $f(\theta)$ is the characteristic function of a measurable set $E$, the existence of the function $\tilde{f}(\theta)$ has a certain geometric significance. It shows that in the neighborhood of almost every point of

$E$ that set has a certain symmetry of structure (for otherwise the integral defining $\tilde{f}(\theta)$ would be divergent). This property is not easily deducible from the familiar properties of measurable sets (for example, from the theorem on the points of density), for otherwise we should have a simple real function proof of the existence of $\tilde{f}(\theta)$.

One might argue that the study of the conjugate functions is strictly speaking outside the scope of the theory of Fourier series. But this is not so, and there seems to be a close relation between the behavior of the partial sums $S_n(\theta)$ of a Fourier series and certain conjugate functions. Let us consider instead of $S_n(\theta)$ the modified partial sums $S_n^*(\theta)$ differing from $S_n(\theta)$ in that only half of the last term is taken. Hence

$$S_n^*(\theta) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) + \tfrac{1}{2}(a_n \cos n\theta + b_n \sin n\theta).$$

For $S_n^*(\theta)$ we have a formula similar to the classical Dirichlet formula,

$$S_n^*(\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta + t)\frac{\sin nt}{2 \tan (t/2)}\,dt$$

(TS, p. 21) and it may be written formally

$$\begin{aligned} S_n^*(\theta) &= \frac{\cos n\theta}{\pi}\int_{-\pi}^{\pi} f(\theta + t) \sin n(\theta + t)\tfrac{1}{2} \cot (t/2) \\ &\quad - \frac{\sin n\theta}{\pi}\int_{-\pi}^{\pi} f(\theta + t) \cos n(\theta + t)\tfrac{1}{2} \cot (t/2)dt \\ &= -g_n(\theta) \cos n\theta + h_n(\theta) \sin n\theta, \end{aligned}$$

say, where $g_n(\theta)$ and $h_n(\theta)$ denote functions conjugate to $f(\theta) \sin n\theta$ and $f(\theta) \cos n\theta$. From this formula and from the slightly strengthened form of the theorem of M. Riesz just stated, the following fact (also due to M. Riesz) follows easily: For every function $f(\theta)$ of the class $L^p$, $p>1$,

$$\int_0^{2\pi} |f(\theta) - S_n(\theta)|^p d\theta \to 0 \qquad \text{as } n \to +\infty.$$

It is a curious fact that in order to prove this result from the theory of Fourier series (in the narrow sense of the word) we have to use properties of conjugate functions. No other proof seems to have been discovered so far.

The theory of the functions $H^p$ has an analogue for functions regu-

lar in a half-plane. The latter theory (developed mainly by Hille and Tamarkin [8]) has its applications mostly in the domain of Fourier integrals, and for this reason we omit its discussion here. Also to the domain of Fourier integrals belong the complex methods developed by Paley and Wiener [18].

3. **The method of conformal representation.** We first recall familiar facts. Let $\Delta$ be any domain in the $\zeta$-plane limited, say, by a simple Jordan curve $\Gamma$. There is a function $\zeta=h(z)$ defined and regular in the unit circle

$$(D) \qquad |z|<1$$

and mapping $D$ conformally onto $\Delta$. The function $h(z)$ may be extended continuously to the closed domain $D+C$, where $C$ is the circumference $|z|=1$, and gives a one-one correspondence between $D+C$ and $\Delta+\Gamma$. If we add some normalizing conditions, the function $h(z)$ is unique.

Let us assume from now on that the curve $\Gamma$ is rectifiable. For point sets situated on a rectifiable curve we have, of course, a theory of measure analogous to that of Lebesgue. In particular, we may speak of sets of measure 0, of "almost everywhere," and so on.

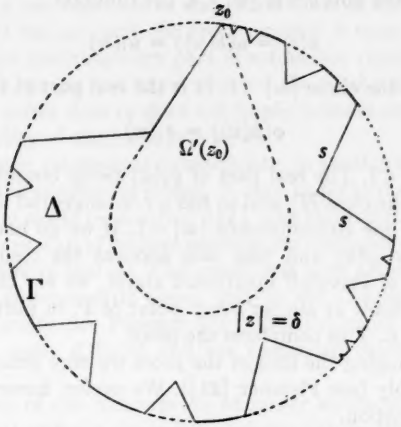
In our case, since $\Gamma$ is of finite length, the function $\phi(z)$ must be of bounded variation on the circumference $C$. This means, as we know, that $\phi(z)$ must be absolutely continuous on $C$. In particular, it transforms every set of measure 0 on $C$ into a set of measure 0 on $\Gamma$. The converse is also true, the sets of measure 0 on $\Gamma$ correspond to sets of measure 0 on $C$. In other words, in our case, the sets of measure 0 on the boundaries are invariants of conformal mapping. This result was first proved by F. and M. Riesz [27].

There is another property, obtained by Privaloff [24], of the mapping function $h(z)$: the mapping is conformal at almost every point of the boundary. This is to be understood as follows. If we exclude a certain point set of measure 0 on $C$, each of the remaining points $z_0$ has the property that if $C'$ and $C''$ are any two paths leading from the interior of $D$ to the point $z_0$ and making angle $\alpha$, the images $\Gamma'$, $\Gamma''$ of $C'$, $C''$ make angle $\alpha$ at the point $\zeta_0=\phi(z_0)$.

In order to see how the above results can be applied, let us sketch the proof of a theorem of Privaloff [23]. Suppose that a function $\phi(z)=u+iv$ is regular in $|z|<1$, and that there is a point set $E$ on the circumference $|z|=1$ with the following property: for every $z_0=e^{i\theta_0}$ belonging to $E$ there is a circular sector $\Omega(z_0)$ with vertex at $z_0$, lying except for $z_0$ entirely inside $C$, and such that the real part $u(z)$ of

$\phi(z)$ is bounded in $\Omega(z_0)$. Then, *the function $\phi(z)$ has a nontangential limit at almost every point $z_0$ of $E$.*

A special case of this result worth a separate statement is: *if the real part $u(z)$ of a regular function $\phi(z)$, $|z|<1$, has a nontangential limit at every point $z_0$ of a set $E$ of the circumference $|z|=1$, the imaginary part $v(z)$ has the same property at almost every point of $E$.*




Let us assume, to fix the ideas, that $\Omega(z_0)$ is symmetrical with respect to the radius $(0, z_0)$. Without loss of generality we may also assume that the function $u(z)$ is bounded *uniformly* in all the $\Omega(z_0)$, $z_0 \in E$. For the upper bound $M(z_0)$ of $u$ in $\Omega(z_0)$ is a function of $z_0$ and is finite on $E$, so it follows that if we reject from $E$ a subset of arbitrarily small measure, $M(z_0)$ will be bounded on the rest of $E$. By a similar argument we may assume that the angles of the sectors $\Omega(z_0)$ are the same, or even that all these sectors are congruent. Let $|z|=\delta$ be the circle tangent to the rectilinear sides (or their continuations) of the sectors $\Omega(z_0)$. Without loss of generality, we may replace each $\Omega(z_0)$ by the domain $\Omega'(z_0)$ limited by the two tangents from $z_0$ to $|z|=\delta$, and by the more distant arc of the circle. One more remark: we may assume that $E$ is a perfect set.

Let now $\Delta$ denote the sum of all the domains $\Omega'(z_0)$ for $z_0 \in E$; $\Delta$ has a starlike shape (see the figure). It is connected. The points of its frontier $\Gamma$, which is a simple Jordan curve, may belong either (a) to the set $E$, (b) to a number of arcs (nonexistent on the figure)

of the circle $|z| = \delta$, (c) to the denumerable set of segments $s$ belonging to the boundaries of some $\Omega'(z_0)$. Since the regions $\Omega'(z_0)$ are all congruent, it is clear that the total length of the segments $s$ does not exceed a fixed multiple (depending only on the shape of the $\Omega'$s) of the total length of the intervals contiguous to $E$. Thus the curve $\Gamma$ is rectifiable.

The function $u(z)$ is bounded in $\Delta$. Hence, if $z=h(w)$ maps $\Delta$ conformally onto the unit circle $|w| < 1$, the function

$$u(z) = u(h(w)) = u_1(w)$$

is harmonic in the circle $|w| < 1$. It is the real part of the function

$$\phi(h(w)) = \phi_1(w)$$

regular in $|w| < 1$. The real part of $\phi_1(w)$ being bounded, the latter function is of the class $H^2$, and so has a nontangential limit at almost every point of the circumference $|w| = 1$. If we go back from $\phi_1(w)$ to the function $\phi(z)$, and take into account the results of F. and M. Riesz and of Privaloff mentioned above, we see that $\phi(z)$ has a nontangential limit at almost every point of $\Gamma$, in particular almost everywhere in $E$. This completes the proof.

Without changing the idea of the proof we may generalize this result considerably (see Plessner [21]). We prefer, however, to give a different application.

Let $\phi(z)$ be a function regular in $|z| < 1$ and of the class $H^2$, and let $\Omega'(z) = \Omega'(\theta)$ have the same meaning as before. Lusin proved that then the integral

$$I(\theta) = \iint_{\Omega'(\theta)} |\phi'(z)|^2 d\omega \tag{22}$$

($d\omega$ an element of area) is finite for almost every $\theta$. This integral represents the area of the domain (generally non-schlicht) obtained from $\Omega'(\theta)$ by the mapping $w=\phi(z)$. By an argument similar to the above we may show (see Marcinkiewicz and Zygmund [15]) that, *if $\phi(z)$ is any function regular in $|z| < 1$, and if it has a nontangential limit at every point $e^{i\theta}$ of a set $E$ of positive measure, then the integral* (22) *is finite almost everywhere in $E$.* Recently, Spencer [29] proved a converse of the above result, namely that *the finiteness of the integral $I(\theta)$ in a set of $\theta$ of positive measure implies the existence of the nontangential limit at almost every point of that set.* Thus we see that almost everywhere the existence of the nontangential limit is equivalent to the finiteness of a certain area.

The method of conformal representation, as presented above, seems to have originated with Golubeff [4]. If we are to use it, we must know the behavior of the function in certain sectorial domains from which we build up a new domain whose boundary has "many" points in common with the boundary of the domain in which the function is defined. It would not work if we wanted, for example, to prove that the existence of the radial limit of $u(z)$ along a set of positive measure of radii implies the existence of the radial limit of $v(z)$ along almost every radius of the set, and the problem itself is open ($u$ and $v$ here are the real and the imaginary part of a function regular in $|z|<1$). In other words, we do not know whether the Abel summability of a trigonometric series does or does not imply (almost everywhere) the Abel summability of the conjugate series.

For some other methods of summability, in particular for ordinary convergence, the problem is solved. A very important step in this direction was first made by Kuttner [12], who showed that *if a Fourier series converges in a set of positive measure, the conjugate series converges almost everywhere in that set.* This result was later on extended to the most general trigonometric series, and to the Cesàro summability. The existing proofs (see Plessner [22], Marcinkiewicz and Zygmund [16, 17]) are quite difficult and are essentially based on complex methods.

The problem of the convergence of power series (or trigonometric series) leads naturally to the problem of the distribution of the partial sums of divergent series. It turns out that under certain conditions the distribution of those partial sums displays a very simple geometric character.

We shall say that a sequence of points $\{s_n\}$ is of *circular structure* with respect to the point $s$, if the derived set of the sequence $\{s_n\}$ consists of a certain number (finite, denumerable, or nondenumerable, but in any event a closed set) of circles with center at the point $s$. If for the sake of simplicity we confine our attention to power series

$$\sum_{\nu=0}^{\infty} c_\nu z^\nu = \phi(z) \tag{23}$$

with bounded coefficients, we have the following result (Marcinkiewicz and Zygmund [17]).

*Suppose that*

$$\lim_{r\to 1} \phi(re^{i\theta}) = s(\theta)$$

*exists for every $\theta$ of a set $E$ of positive measure. Then, for almost every $\theta$ of $E$, the sequence $\{s_\nu(e^{i\theta})\}$ of the partial sums of the series (23) is of circular structure with respect to the point $s(\theta)$. In particular, if the coefficients $c_\nu$ tend to 0, then for almost every $\theta \in E$ the derived set of the sequence $\{s_\nu(e^{i\theta})\}$ is a circle (finite or infinite) with center at the point $s(\theta)$.*

Quite recently new applications of conformal mapping to the theory of trigonometric series were obtained by F. Wolf. He combined conformal mapping with certain extensions of the Phragmèn-Lindelöf principle. I would like to mention here one of the important results he obtained.

The problem whether a given function can be represented by more than one trigonometric series (not necessarily a Fourier series) may be reduced, by subtracting these series, to the following problem: can a trigonometric series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \tag{24}$$

which does not vanish identically represent zero? This "problem of uniqueness" of trigonometric series has various aspects. It is a classical fact that in the case of convergent series it admits of a positive solution: if the series (24) converges to 0 at every point, the coefficients $a_\nu$ and $b_\nu$ all vanish. The next step is to consider summable series. The case of Abel summability is of particular importance since it means studying the radial behavior of the harmonic function

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) r^\nu \tag{25}$$

associated with the series (24). Rajchman [25] proved that if the function (25) tends to 0 along every radius and if in addition $|a_\nu| + |b_\nu| \to 0$, the series (24) vanishes identically. Verblunsky [31] replaced the last condition by $|a_\nu| + |b_\nu| = O(\nu)$. If we consider only radial limits, we cannot go much further since there are trigonometric series with coefficients $O(\nu)$, summable $A$ everywhere, and yet not vanishing identically: the series

$$\sum_{\nu=1}^{\infty} \nu \sin \nu\theta$$

is an instance in point. Thus we must impose new restrictions if we wish to obtain positive results. Wolf [32] proved that:

*If* (i) $u(r, \theta)$ *is harmonic for* $r<1$, (ii) *for every* $\theta_0$, $u(r, \theta)\to 0$ *as* $(r, \theta)\to(1, \theta_0)$ *along any nontangential path*, (iii) $|u(r, \theta)| \leqq \exp A/(1-r)^m$, *where* $A$ *and* $m$ *are independent of* $r$ *and* $\theta$, *then the series* (24) *vanishes identically.*

4. **The Littlewood-Paley method.** This method (see Littlewood and Paley [13]) cannot be easily described without going into technical details, and by necessity I shall have to be rather brief here. Let me start with some of the results they achieved. Suppose that $f(\theta)$ is an integrable function of period $2\pi$, and let $S_n(\theta)$ denote the $n$th partial sum of the Fourier series of $f(\theta)$. It is a familiar fact that $S_n(\theta)$ may diverge at some points even if $f(\theta)$ is a continuous function. Whether there exists a continuous function $f$ such that $S_n(\theta)$ diverges everywhere, or at least almost everywhere, is still an open problem. For $f$ merely integrable, the divergence of $S_n(\theta)$ may actually occur everywhere, as shown by Kolmogoroff [9; TS, p. 175]. It is a curious fact that so far it has not been possible to construct a similar example for functions of any class $L^p$, $p>1$, and the problem seems to be much more difficult there.

Of course, from a certain point of view the function $f(\theta)$ is adequately represented by the Fejér means of its Fourier series, but the behavior of the partial sums $S_n$ is of considerable intrinsic interest. For functions of the class $L^p$, $p>1$, Paley and Littlewood proved the following result (the special case $p=2$ was solved earlier by Kolmogoroff [10; TS, p. 257]).

*Whatever the sequence of positive integers* $n_1, n_2, \cdots$ *satisfying an inequality*

$$n_{k+1}/n_k > q > 1,$$

*the partial sums* $S_{n_k}(\theta)$ *converge to* $f(\theta)$ *at almost every point.*

The result is false for $p=1$ (although it holds for power series of the class $H$; see Zygmund [34]). Its significance consists in the fact that the sequence $n_k$ is independent of the function $f$.

Another result which Littlewood and Paley proved concerns the *convergence factors* of Fourier series. It is a familiar fact due to Hardy that if (24) is the Fourier series of an integrable function, the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/\log \nu$$

converges for almost every $\theta$. If $f$ is of integrable square, even the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/(\log \nu)^{1/2} \tag{26}$$

converges almost everywhere (Kolmogoroff and Seliverstoff [11], Plessner [20]; TS, p. 252). Littlewood and Paley proved the intermediate result: if $f \in L^p$, $1 < p < 2$, the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/(\log \nu)^{1/p}$$

converges almost everywhere. Nothing is known about the case $p > 2$. In particular, about convergence factors for continuous functions we know no more than for functions of the class $L^2$, that is to say that the series (26) is convergent almost everywhere. The latter fact is equivalent to saying that the partial sums $S_n(\theta)$ of the Fourier series satisfy the relation

$$S_n(\theta) = o(\log n)^{1/2} \tag{27}$$

at almost every point. It may be that the estimate (27) is the best result for the Fourier series of continuous functions.

The Littlewood-Paley theory is based on the use of the function $g(\theta)$ defined by the formula

$$g(\theta) = \left( \int_0^1 (1 - r) \left| \phi'(re^{i\theta}) \right|^2 dr \right)^{1/2}$$

where $\phi(z)$ is any function regular in $|z| < 1$. The function $g(\theta)$ has no simple geometric interpretation, but has some connection with the integral $I(\theta)$ (see (22)) whose geometric significance and relation to the existence of the nontangential limit was already mentioned. Let

$$s(\theta) = I^{1/2}(\theta),$$

so that $s(\theta)$ is of dimension 1. It may be shown that $s(\theta)$ is, effectively, a majorant of $g(\theta)$; more precisely, $g(\theta) \leqq C_\alpha s(\theta)$ where $C_\alpha$ depends only on the angle $\alpha$. The results obtained by Littlewood and Paley for the function $g(\theta)$ when interpreted in terms of the function $s(\theta)$ may be stated as follows: *If* $\phi(z) \in H^p$, $p > 1$, *then*

$$\int_0^{2\pi} s^p(\theta) d\theta \leqq A_{p,\alpha} \int_0^{2\pi} \left| \phi(e^{i\theta}) \right|^p d\theta,$$

$$\int_0^{2\pi} \left| \phi(e^{i\theta}) \right|^p d\theta \leqq B_{p,\alpha} \int_0^{2\pi} s^p(\theta) d\theta.$$

Here $A_{p,\alpha}$ and $B_{p,\alpha}$ depend on $p$ and $\alpha$ only; in addition, for the validity of the second inequality we have to assume that $\phi(0)=0$. Thus the integrability of $s(\theta)$ imitates that of $|\phi(e^{i\theta})|$.

## References


1. A. S. Besicovich, *On a general metric property of summable functions*, J. London Math. Soc. vol. 1 (1926) pp. 120–128.

2. G. C. Evans, *The logarithmic potential*, Amer. Math. Soc. Colloquium Publications, vol. 6, New York, 1927.

3. P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. vol. 30 (1906) pp. 305–400.

4. V. V. Golubeff, *Sur les fonctions analytiques uniformes admettant un ensemble parfait de points singuliers*, Moskva, Zapiski Universiteta, Otdiel fisiko-matematicheskii vol. 29 (1916).

5. G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*, Math. Ann. vol. 97 (1926) pp. 159–209.

6. ———, *A maximal theorem with function-theoretic applications*, Acta Math. vol. 54 (1930) pp. 81–116.

7. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1934.

8. E. Hille and J. D. Tamarkin, *On the absolute integrability of Fourier transforms*, Fund. Math. vol. 25 (1935) pp. 329–352.

9. A. Kolmogoroff, *Une série de Fourier-Lebesgue divergente partout*, C. R. Acad. Sci. Paris vol. 183 (1926) pp. 1327–1328.

10. ———, *Une contribution à l'étude de la convergence des séries de Fourier*, Fund. Math. vol. 5 (1924) pp. 96–97.

11. A. Kolmogoroff and G. Seliverstoff, *Sur la convergence des séries de Fourier*, Rendiconti della Accademia dei Lincei (6) vol. 3 (1926) pp. 307–310.

12. B. Kuttner, *A theorem on trigonometric series*, J. London Math. Soc. vol. 10 (1935) pp. 131–140.

13. J. E. Littlewood and R. E. A. C. Paley, *Theorems on power series and Fourier series*. I, II, III, J. London Math. Soc. vol. 6 (1931) pp. 230–233; Proc. London Math. Soc. vol. 42 (1937) pp. 52–89; ibid. vol. 43 (1937) pp. 105–126.

14. J. Marcinkiewicz, *Sur les séries de Fourier*, Fund. Math. vol. 27 (1936) pp. 38–69.

15. J. Marcinkiewicz and A. Zygmund, *A theorem of Lusin*, Duke Math. J. vol. 4 (1938) pp. 473–485.

16. ———, *On the differentiability of functions and the summability of trigonometric series*, Fund. Math. vol. 26 (1936) pp. 1–43.

17. ———, *On the behavior of trigonometric series and power series*, Trans. Amer. Math. Soc. vol. 50 (1941) pp. 407–453.

18. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, New York, 1934.

19. A. Plessner, *Zur Theorie der konjugierten trigonometrischen Reihen*, Mitteilungen des Mathematische Seminar, Universität Giessen, 1923.

20. ———, *Über Konvergenz von trigonometrischen Reihen*, J. Reine Angew. Math. vol. 155 (1926) pp. 15–25.

21. ———, *Über das Verhalten analytischer Funktionen auf dem Rande des Definitionsbereiches*, ibid. vol. 158 (1928) pp. 219–227.

22. ———, *Über konjugierte trigonometrische Reihen*, C. R. (Doklady) Acad. Sci. URSS. vol. 4 (1935) pp. 251–253.

23. I. Privaloff, *Sur une généralisation du théorème de Fatou*, Rec. Math. (Mat. Sbornik) vol. 31 (1923) pp. 232–235.

24. ———, *Intégrale de Cauchy* (in Russian), Saratoff, 1919, pp. 1–94.

25. A. Rajchman, *Sur les séries trigonométriques sommables par le procédé de Poisson*, Prace Matematyczno-Fizyczne.

26. F. Riesz, *Über die Randwerte einer analytischer Funktion*, Math. Zeit. vol. 18 (1922) pp. 87–95.

27. F. and M. Riesz, *Über Randwerte einer analytischen Funktion*, Quatrième Congres des mathématiciens scandinaves, 1916, pp. 27–44.

28. M. Riesz, *Sur les séries conjuguées*, Math. Zeit. vol. 27 (1927) pp. 218–244.

29. D. C. Spencer, *A function-theoretic identity*, Amer. J. Math.

30. E. C. Titchmarsh, *On conjugate functions*, Proc. London Math. Soc. vol. 29 (1928) pp. 49–80.

31. S. Verblunsky, *On the theory of trigonometric series*. I, Proc. London Math. Soc. vol. 34 (1932) pp. 441–456.

32. F. Wolf, *The Poisson integral*, Acta Math. vol. 74 (1941) pp. 65–100.

33. A. Zygmund, *Trigonometric series*, Warsaw, 1935 (Stechert, New York).

34. ———, *On the convergence and summability of power series on the circle of convergence*. I, Fund. Math. vol. 30 (1938) pp. 170–196.

Mt. Holyoke College

## THE SUMMER MEETING IN NEW BRUNSWICK

The forty-ninth Summer Meeting of the Society and the twenty-fifth Colloquium were held at New Jersey College for Women, Rutgers University, on Sunday and Monday, September 12–13, 1943. The Mathematical Association of America met on Saturday, September 11, and the Institute of Mathematical Statistics met on Sunday and jointly with the Society on Monday morning. About three hundred fifty persons attended the meetings among whom were the following two hundred forty members of the Society:

C. R. Adams, E. B. Allen, C. B. Allendoerfer, Warren Ambrose, T. W. Anderson, H. E. Arnold, K. J. Arnold, L. A. Aroian, W. L. Ayres, A. V. Baez, F. E. Baker, D. H. Ballou, Valentin Bargmann, J. L. Barnes, P. T. Bateman, B. M. Bennett, Stefan Bergman, Lipman Bers, J. H. Bigelow, Garrett Birkhoff, G. D. Birkhoff, B. H. Bissinger, E. E. Blanche, W. A. Blankinship, E. M. Boggs, T. A. Botts, J. W. Bower, A. H. Bowker, S. E. Brasefield, H. W. Brinkmann, B. H. Brown, C. T. Bumer, L. H. Bunyan, Herbert Busemann, Hobart Bushey, J. H. Bushey, S. S. Cairns, J. W. Calkin, Achille Capecelatro, M. E. Carlen, W. B. Carver, C. R. Cassity, Abraham Charnes, M. D. Clement, A. B. Coble, H. R. Cooley, T. F. Cope, Byron Cosby, Richard Courant, A. P. Cowgill, C. C. Craig, E. L. Crow, H. B. Curry, J. H. Curtiss, E. H. Cutler, J. F. Daly, D. R. Davis, F. F. Decker, C. L. Dolph, Arnold Dresden, R. F. Dressler, William H. Durfee, M. L. Elveback, Benjamin Epstein, Paul Erdös, G. M. Ewing, W. K. Feller, F. G. Fender, F. A. Ficken, W. W. Flexner, Tomlinson Fort, R. M. Foster, F. H. Fowler, J. S. Frame, Hans Fried, Bernard Friedman, K. O. Friedrichs, Orrin Frink, R. E. Fullerton, A. S. Galbraith, G. N. Garrison, R. E. Gaskell, Abe Gelbart, B. P. Gill, S. H. Gould, H. S. Grant, J. W. Green, Lewis Greenwald, C. C. Grove, V. G. Grove, E. J. Gumbel, Theodore Hailperin, D. W. Hall, P. R. Halmos, G. E. Hay, G. A. Hedlund, M. H. Heins, Aaron Herschfeld, Fritz Herzog, T. H. Hildebrandt, Einar Hille, T. R. Hollcroft, Harold Hotelling, S. E. Hotelling, E. M. Hull, Witold Hurewicz, H. D. Huskey, L. C. Hutchinson, R. E. Johnson, B. W. Jones, Edward Kasner, R. B. Kershner, J. R. Kline, E. G. Kogbetliantz, Arthur Korn, W. D. Lambert, K. W. Lamson, H. G. Landau, V. V. Latshaw, Solomon Lefschetz, Joseph Lehner, Marguerite Lehr, D. C. Lewis, Marie Litzinger, J. N. B. Livingood, Simon Lopata, E. R. Lorch, R. T. Luginbuhl, P. H. McGrath, E. J. McShane, C. C. MacDuffee, G. W. Mackey, Saunders MacLane, H. F. MacNeish, W. G. Madow, M. P. Martin, W. T. Martin, A. E. Meder, F. H. Miller, K. S. Miller, L. W. Miller, E. B. Mode, Deane Montgomery, R. K. Morley, Richard Morris, D. S. Morse, E. J. Moulton, F. D. Murnaghan, W. R. Murray, C. A. Nelson, P. B. Norman, C. O. Oakley, L. F. Ollmann, P. S. Olmstead, F. W. Owens, Gordon Pall, C. W. Pflaum, R. S. Phillips, Everett Pitcher, Hillel Poritsky, Willy Prager, M. H. Protter, Hans Rademacher, J. F. Randolph, C. H. Rawlins, G. E. Raynor, O. H. Rechard, C. J. Rees, Mina Rees, C. F. Rehberg, R. G. D. Richardson, D. E. Richmond, C. E. Rickart, J. F. Ritt, H. E. Robbins, M. S. Robertson, R. E. Root, J. H. Rosenbloom, P. C. Rosenbloom, A. E. Ross, J. B. Rosser, S. G. Roth, Raphael Salem, Hans Samelson, F. E. Satterthwaite, R. H. Scanlan, Henry Scheffé, S. A. Schelkunoff, I. J. Schoenberg, Abraham Schwartz, H. M. Schwartz, G. E. Schweigert, Seymour Sherman, Max Shiffman, J. A. Shohat,

D. T. Sigley, L. L. Smail, D. M. Smiley, M. F. Smiley, T. L. Smith, W. M. Smith, Ernst Snapper, Andrew Sobczyk, R. D. Specht, E. P. Starke, H. W. Steinhaus, R. W. Stokes, R. R. Stoll, R. E. Street, R. C. Strodt, W. C. Strodt, Ethel Sutherland, J. L. Synge, Otto Szász, J. D. Tamarkin, J. I. Tracey, C. A. Truesdell, H. F. Tuan, A. W. Tucker, J. W. Tukey, D. F. Votaw, Abraham Wald, G. L. Walker, R. J. Walker, A. D. Wallace, R. M. Walter, G. C. Webber, K. W. Wegner, A. P. Wheeler, P. M. Whitman, Hassler Whitney, G. T. Whyburn, J. E. Wilkins, S. S. Wilks, W. L. G. Williams, Jacob Wolfowitz, Bertram Yood, J. W. T. Youngs, Oscar Zariski, Antoni Zygmund.

The Colloquium Lectures on the subject *Existence theorems in the calculus of variations* were delivered by Professor E. J. McShane of the University of Virginia on Sunday morning and Monday morning and afternoon. Acting President C. C. MacDuffee presided at the first and third lectures and Vice President J. D. Tamarkin at the second.

On Sunday at 2:00 P.M., Professor Antoni Zygmund of Mount Holyoke College gave an address entitled *The complex method of the theory of trigonometric series*. Professor G. D. Birkhoff presided.

At 3:15 P.M., Sunday, Professor F. D. Murnaghan of Johns Hopkins University gave an address entitled *Finite deformations of an elastic solid*. The discussion was led by Professor J. L. Synge of Ohio State University. Dean R. G. D. Richardson presided.

There was a business meeting Monday morning at which the Society voted to suspend for the duration of the war the first sentence of Article VIII, Section 1, of the By-Laws which reads "The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following."

The presiding officers for the sessions of short papers were: Applied Mathematics, Sunday morning, Professor Hobart Bushey; Analysis, Sunday morning, Professor Hassler Whitney; Probability and Statistics (a joint session with the Institute of Mathematical Statistics), Monday morning, Professor C. C. Craig; Algebra, Geometry, and Topology, Monday morning, Professor G. A. Hedlund; Applied Mathematics, Monday afternoon, Professor F. D. Murnaghan; Analysis and Logic, Monday afternoon, Professor J. B. Rosser.

Headquarters for the meeting were in the Little Arcade on Jameson Campus. Rooms in the dormitories were made available to those attending the meetings.

On Saturday afternoon Dean Margaret T. Corwin received the mathematicians and their guests at her home.

A photograph was taken at 1:15 P.M., Sunday.

On Sunday evening there was a dinner for members of the Society, the Association, and the Institute and their guests. Professor Richard Morris was toastmaster. Dean Corwin welcomed the guests on behalf

of President R. C. Clothier of Rutgers University. Dean R. G. D. Richardson of Brown University spoke of the importance of applied mathematics in the war effort. Professor G. D. Birkhoff of Harvard University urged mathematicians and scientists to maintain a proper balance of values during the emergency. Professor W. L. Ayres of Purdue University presented a resolution of thanks to the President and Dean of New Jersey College for Women of Rutgers University, the local committee and all who assisted them for their excellent arrangements and cordial hospitality.

Following the dinner, members of the Department of Music of New Jersey College for Women gave an excellent musicale.

The Council met on Sunday, September 12, at 9:00 P.M. in the lounge at Jameson Campus.

The Secretary announced the election of the following twenty-eight persons to membership in the Society:

Professor Ernest Willard Anderson, Iowa State College;
Dr. Melvin Avrami, Columbia University;
Mr. Ralph Herbert Beard, New York Telephone Company;
Mr. J. Oliver Brown, Hampton Institute, Hampton, Va.;
Dr. Wei-Zang Chien, New York, N. Y.;
Mr. Murray Ellis, University of Chicago;
Mr. Eugene Fately, U. S. Army, Camp Murphy, Fla.;
Dr. Casper Goffman, Westinghouse Electric and Manufacturing Company, Pittsburgh, Pa.;
Mr. Clarence Frederic Hall, Rensselaer Polytechnic Institute;
Dr. Robert Griffith Helsel, Ohio State University;
Professor Gerald Boone Huff, Southern Methodist University;
Mr. Solomon Hurwitz, Brooklyn College;
Mrs. Florence Jacobson, University of North Carolina;
Mr. Robert Kates, University of Chicago;
Dr. Ervand G. Kogbetliantz, Lehigh University;
Dr. Zbigniew Lepecki, University of Paraná, Paraná, Brazil;
Mr. Simon Lopata, Columbia University;
Mr. Elbert D. Miller, Yuba Junior College, Marysville, Calif.;
Mr. Kenneth S. Miller, Yonkers, N. Y.;
Dr. Alfonso Gándara Nápoles, Institute of Mathematics, National University of Mexico;
Dr. Paul Felix Nemenyi, University of Colorado;
Dr. Leonard Nicholl, Kay-Fries Chemicals, West Haverstraw, N. Y.;
Mr. Robinson Henry Parson, Houston College for Negroes;
Mr. George Paul Roland, Department of Public Works, Albany, N. Y.;
Dr. Fritz Rothberger, Acadia University;
Dr. Feodor Theilheimer, Trinity College, Hartford, Conn.;
Mr. Milton Albert Treuhaft, Westinghouse Radio Division, Baltimore, Md.;
Professor Wyman Loren Williams, University of South Carolina.

The following appointments by the President were reported: as

representative of the Society at the meeting in tribute to Nicholas Copernicus in New York City on May 24, 1943, Professor C. C. MacDuffee; as a Committee on Arrangements for the Annual Meeting in Chicago on November 26–27, 1943, Professors L. R. Ford (Chairman), W. L. Ayres, W. B. Carver, A. S. Householder, R. G. Sanger, and H. S. Wall; as liaison officer of the Society in connection with the editorial management of the Quarterly of Applied Mathematics, Professor Einar Hille; as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings, to replace Professor M. H. Stone during his absence from the country, Professor J. F. Ritt.

It was announced that a report was sent in July to the Carnegie Corporation regarding the grant to Mathematical Reviews. Attention was called to the fact that the number of Mathematical Reviews subscribers for 1943 is 1239. This subscription list is undoubtedly larger than that of any mathematical journal which is not provided by a mathematical society to its members as one of the privileges of membership.

Certain invitations to give invited addresses were announced: Professor Reinhold Baer for the 1943 annual meeting in Chicago; Dr. Alfred Tarski, Professor M. A. Zorn, and Professor Theodore von Kármán for the November, 1943, meeting in Pasadena; Dr. D. H. Hyers and Professor R. M. Robinson for the 1944 spring meeting in Berkeley; Professors J. L. Synge and I. S. Sokolnikoff for the 1944 spring meeting in Chicago.

It was reported that fifteen microfilm reading machines had been sent to the London Mathematical Society for distribution to scientific organizations in Great Britain. It was also reported that a microfilm reading machine and copies of the Semicentennial volumes had been donated to the library of the American Philosophical Society.

On recommendation of the Colloquium Editorial Committee, the Council voted to invite Professor Tibor Radó to deliver a series of Colloquium Lectures in 1945.

Dr. Warren Weaver was appointed representative on the National Research Council for a period of three years beginning July, 1944.

Professor E. J. Moulton and Dean W. G. Simon were reappointed representatives on the Council of the American Association for the Advancement of Science for 1944.

Dean R. G. D. Richardson was appointed as an additional member of the Committee on Addresses in Applied Mathematics. The other members of this committee are Professor Richard Courant (Chairman), Mr. R. M. Foster, and Professor Harold Hotelling.

The Secretary reported the foundation of the Mexican Mathematical Society on June 30, 1943. (The inaugural ceremonies were originally planned for April 9, 1943, as reported on page 517 of the July Bulletin.) Professor Alfonso Nápoles Gándara is the president of the new organization. Greetings were sent to the group on behalf of the American Mathematical Society.

The Council adopted resolutions on the deaths of the following persons who have been prominent in the work of the Society:

### HOWARD HAWKES MITCHELL

The Council of the American Mathematical Society records its deep sense of loss in the death on March 13, 1943, of Howard Hawkes Mitchell, Professor of Mathematics at the University of Pennsylvania.

Professor Mitchell was prominent in all phases of the Society's activity. He was a member of the Council from 1920 to 1922 and Vice President from 1932 to 1933. He was made an Associate Editor of the Transactions of the Society in 1920 and in 1925 was elected to the Editorial Committee of that journal. He served as a member of this Committee with distinction until 1930. His keen insight and excellent mathematical intuition fitted him particularly for the exacting duties of an Editor and, in every way, he maintained the high standards of the Transactions. He was most sympathetic in the handling of the work of younger mathematicians and was a valuable source of inspiration to them for their further research.

He was a kindly gentleman with broad interests and high ideals. In every manner he exemplified the finest in American mathematics and it is with deep emotion that we present this expression of our feelings on his passing.

### WILLIAM FOGG OSGOOD

William Fogg Osgood, thoroughly representative of the mathematicians of his time, was self-disciplined in method, painstaking in detail, rigorous and clear in arriving at his results. For him mathematics was a way of discovery, and each individual function was an object worth knowing in all of its details. The method of strict analysis was a keen tool whose fineness he reverenced, and no labor was too great for him if he could create an end product that was smoothly machined in every part. At a time when scholarship and technique were of the highest repute, he was of the best of artist craftsmen. Some of his ideas were instrumental in forging new concepts. Yet he was innately conservative, because he felt that he had at his command all the tools which he needed in investigating those things which were vital to him.

He had his share in building the Harvard Division of Mathematics to an outstanding world position in teaching and research. But perhaps it is as a teacher of the first course in calculus that most of his pupils remember him. His classes in Mathematics 2 contained men of all ages, from the young freshman or sophomore who was stepping abroad in mathematics for the first time, to the experienced teacher who spent his sabbatical leave in learning how to solve the teaching difficulties which had plagued him. Osgood carried his mathematical straightforwardness fully into his instruction. His aim was to teach ideas, and to make clear the difficult ones, so that they might serve as a basis of further knowledge. The way of mathematics is long and difficult and makes great demands on will and conscience. Osgood wanted no false starts. His purpose sank deep into his students' minds, and they have never ceased to be grateful.

## CLARA ELIZA SMITH

The Council of the American Mathematical Society desires to record its grateful appreciation of the service of Clara Eliza Smith, Professor of Mathematics at Wellesley College, deceased on May 12, 1943. Professor Smith's devotion to the Society was such that in the dozen years beginning with 1922, when it was necessary greatly to increase the membership of the Society in order to find financial support for its publications, she labored faithfully and unobtrusively as a member of the Committee on Membership, doubling the roll of members. Much of the success of that committee was due to her laborious and painstaking assembling of data regarding teachers of college mathematics. On behalf of the Society, the Council wishes to acknowledge her loving interest in its welfare.

## EDWARD BURR VAN VLECK

June 7, 1863–June 2, 1943

The American Mathematical Society notes with sorrow the death of Professor E. B. Van Vleck, whom it honored as an erstwhile active participant and leader in its affairs, and as a noted mathematical scholar and investigator.

Professor Van Vleck's membership in this Society extended over more than a half century, dating from the year 1893, when he was fresh from the attainment of his doctorate. Over the course of many years he was a frequent attendant at the Society's meetings, and marked many of these occasions by the presentations of results of his investigations. The mathematical literature of those years is dotted with papers of his on infinite series, on functions defined by ordinary differential equations, on continued fractions, on point sets, on functional equations, on the roots of polynomials and on other subjects.

In recognition of his productive mathematical work Professor Van Vleck was variously and highly honored by this Society. The posts which were bestowed upon him include those of: Member of the Council, 1902–1904; Colloquium lecturer (The Boston Colloquium), 1903; Editor of the Transactions, 1905–1910; President, 1913–1914. Honors bestowed upon him by other sources include the Doctorate of Laws (Clark University), 1909; Election to the National Academy, 1911; the chairmanship of Section A of the American Association for the Advancement of Science, 1912; the Doctorate of Mathematics and Physics (University of Groningen), 1914; the Doctorate of Science (University of Chicago), 1916; Decoration by the French Government as "Officer de l'instruction publique," 1920; the Doctorate of Laws (Wesleyan University), 1925.

In his person Professor Van Vleck was modest and kindly. He was a sensitive man of broad cultural interests. As a teacher he stood on all occasions and steadfastly for the highest standards of achievement and of scholarly integrity.

The American Mathematical Society is indebted to Professor Van Vleck for his participation and counsel, which came to it during those important years when its membership was to be counted by hundreds rather than the present day's thousands. It gratefully acknowledges this, and mourns his departure from this life.

## HENRY SEELY WHITE

Inasmuch as Henry Seely White, ninth President of the Society, 1903–1904, died on May 20, 1943; and

Inasmuch as he was very much one of a small number of enthusiastic young Americans, trained in Europe, who, near the close of the last century, determined to

give their lives to bringing American mathematical study and research up to the standard attained in Europe; and

Inasmuch as a most necessary element in accomplishing this purpose was founding and cherishing our Society, guiding its meetings and caring for its publications; and

Inasmuch as he was an unselfish and helpful friend to all who came to him for counsel or encouragement, were they contemporaries or youthful pupils; and

Inasmuch as his interest in mathematics never slackened, so that his most sustained piece of writing, a treatise on cubic curves, appeared near the close of his career; be it therefore

Resolved that the Society hereby place on record its deep sense of loss in his death.

The Council passed unanimously the following resolution: that it is the sense of the Council that the roster of scientific meetings of the Society should be continued at at least the same level as in preceding years. The Council designated the November 26–27, 1943, meeting in Chicago as the Annual Meeting of the Society. At this meeting the Bôcher Prize for papers in analysis published during the period of 1938–1942 will be awarded and Professor Marston Morse will deliver his retiring presidential address on *New settings for topology in analysis*. Times and places of meetings during 1944 were set as follows: February 26–27 in New York City; October 28 in New York City.

The committee on Mathematical Surveys reported the publication of the two monographs, *The theory of rings* by Professor Nathan Jacobson and *The problem of moments* by Professors J. A. Shohat and J. D. Tamarkin. The Council voted to continue the new series by publishing other books of the same general character as these two monographs.

Professor MacDuffee, Acting Chairman of the War Policy Committee, reported for the committee the following items: (1) Professors Stone and Morse have been representing the American Mathematical Society and the Mathematical Association of America in Washington on matters affecting the use of mathematically-trained and qualified personnel, working in close cooperation with Dean H. L. Dodge, Director of the Office of Scientific Personnel. (2) The Rockefeller Foundation has made a grant of $2500 to defray the expenses of the War Policy Committee. Of this amount $1,000 has been appropriated to the support of the Office of Scientific Personnel. (3) A subcommittee of the War Policy Committee to gather information as to the actual effect of the mathematical aspects of the Army and Navy training programs, and to formulate suggestions for their improvement, has been established. Professor W. L. Hart is Chairman and the other members are Professors C. R. Adams, H. M. Bacon, B. H. Brown, H. J. Ettlinger, C. V. Newsom, G. B. Price, and W. M. Whyburn.

(4) The War Policy Committee has recognized the desirability of gathering information concerning the war activities of mathematicians but at present sees no way of carrying out a plan for doing so. Members of the Society and Association are urged to cooperate in making this undertaking possible at a later date. By keeping a careful and detailed record of his own activities and experience during the war period, each mathematician will contribute to the preservation of the data without which a record of this extraordinarily significant period in our history would be impossible. (5) The Committee on Available Teachers of Collegiate Mathematics has been functioning since April, 1943. A report on the activities of this Committee appears elsewhere in this Bulletin.

Titles and cross references to the abstracts of papers read are given below. The papers were read as follows: papers 1–8 in the section for Applied Mathematics on Sunday morning; papers 9–15 in the section for Analysis on Sunday morning; papers 16–20 in the section for Probability and Statistics on Monday morning; papers 21–26 in the section for Algebra, Geometry, and Topology on Monday morning; papers 27–32 in the section for Applied Mathematics on Monday afternoon; papers 33–36 in the section for Analysis and Logic on Monday afternoon; and papers 37–88, whose abstract numbers are followed by the letter *t*, were read by title. Professor Lee was introduced by Professor Leonard Carlitz, Mr. Handelman by Professor Willy Prager, Dr. Merrill by Professor E. B. Allen, Mr. Hsiung and Mr. Wang by Professor J. A. Shohat, and Mr. Salzer by Professor A. N. Lowan. Paper 3 was presented by Professor Hay, paper 7 by both authors, paper 11 by Dr. Herzog, paper 12 by Professor Martin, paper 14 by Dr. Fried, papers 19 and 20 by Professor Wald, and paper 24 by Professor Pall.

1. L. C. Hutchinson: *Free vibrations in a rectangular rod.* Preliminary report. (Abstract 49-11-284.)

2. E. G. Kogbetliantz: *Detailed quantitative interpretation of maps of gravitational anomalies with the aid of mathematical analysis.* Preliminary report. (Abstract 49 9-223.)

3. G. E. Hay and Willy Prager: *On plane rigid frames containing curved members.* (Abstract 49-9-221.)

4. Garrett Birkhoff: *Reversibility paradox and camber.* (Abstract 49-11-277.)

5. Arthur Korn: *Compressibility with respect to vibrations of high frequency.* (Abstract 49-11-285.)

6. Stefan Bergman: *Solutions of linear partial differential equations of fourth order.* (Abstract 49-9-218.)

7. D. W. Dudley and Hillel Poritsky: *The geometry of cutting and hobbing of worms and gears.* (Abstract 49-11-279.)

8. Otto Szász: *On uniform convergence of Fourier series.* (Abstract 49-9-215.)

9. C. E. Rickart: *An abstract Radon-Nikodym theorem.* (Abstract 49-11-270.)

10. B. H. Bissinger: *Generalizations of continued fractions.* (Abstract 49-11-252.)

11. Fritz Herzog and B. H. Bissinger: *Generalization of Borel's and F. Bernstein's theorems on continued fractions.* (Abstract 49-11-253.)

12. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under translations.* (Abstract 49-9-204.)

13. Tomlinson Fort: *The weighted vibrating string and its limit.* (Abstract 49-9-207.)

14. Paul Erdös and Hans Fried: *On the connection between gaps in power series and the roots of their partial sums.* (Abstract 49-11-258.)

15. P. C. Rosenbloom: *Sequences of polynomials, especially sections of power series.* (Abstract 49-9-210.)

16. Jacob Wolfowitz: *Asymptotic distributions of ascending and descending runs.* (Abstract 49-7-197.)

17. E. J. Gumbel: *On the plotting of statistical observations.* (Abstract 49-9-238.)

18. W. K. Feller: *On a general class of "contagious" distributions.* (Abstract 49-11-301.)

19. Abraham Wald and Jacob Wolfowitz: *An exact test for randomness in the non-parametric case based on serial correlation.* (Abstract 49-11-303.)

20. H. B. Mann and Abraham Wald: *On the statistical treatment of linear stochastic difference equations.* (Abstract 49-11-302.)

21. William H. Durfee: *Congruence of quadratic forms over valuation rings.* (Abstract 49-9-200.)

22. Edward Kasner: *Motion in a resisting medium.* (Abstract 49-9-234.)

23. V. G. Grove: *A general theory of surfaces and conjugate nets.* (Abstract 49-9-230.)

24. R. E. O'Connor and Gordon Pall: *On integral quadratic forms in many variables.* (Abstract 49-11-247.)

25. H. L. Lee: *The sum of the kth power of polynomials of degree m in a Galois field.* Preliminary report. (Abstract 49-9-201.)

26. H. D. Huskey: *On polyhedra and polyhedral path surfaces.* (Abstract 49-9-240.)

27. C. A. Truesdell: *The differential equations of the membrane theory of shells of revolution.* (Abstract 49-11-294.)

28. G. H. Handelman: *On a principle of M. A. Sadowsky.* Preliminary report. (Abstract 49-11-282.)

29. A. E. Ross: *A note on forced oscillations of a conducting sphere.* (Abstract 49-11-292.)

30. S. H. Gould: *The Rayleigh-Ritz method for higher eigenvalues.* (Abstract 49-9-220.)

31. R. E. Gaskell: *On moment balancing in structural dynamics.* (Abstract 49-11-281.)

32. L. L. Merrill: *The mathematical determination of cooling rates during arc welding.* (Abstract 49-11-287.)

33. J. E. Wilkins: *Multiple integral problems in parametric form in the calculus of variations.* (Abstract 49-11-275.)

34. F. A. Ficken: *On two criteria for convexity in a symmetric Banach space.* (Abstract 49-5-134.)

35. Theodore Hailperin: *A set of axioms for logic.* (Abstract 49-9-236.)

36. Andrew Sobczyk: *On the extension of linear transformations.* (Abstract 49-9-214.)

37. A. A. Albert: *Quasigroups.* I. (Abstract 49-9-199-*t*.)

38. E. F. Beckenbach and R. H. Bing: *Conformal minimal varieties.* (Abstract 49-7-182-*t*.)

39. Stefan Bergman: *Fundamental solutions of partial differential equations of the second order.* (Abstract 49-11-251-*t*.)

40. L. M. Blumenthal: *Metric study of generalized elliptic spaces.* Preliminary report. (Abstract 49-11-304-*t*.)

41. L. M. Blumenthal: *New formulations of some imbedding theorems.* (Abstract 49-11-305-*t*.)

42. R. P. Boas: *Almost periodic functions of exponential type.* (Abstract 49-7-183-*t*.)

43. Herbert Busemann: *On spaces in which two points determine a geodesic.* (Abstract 49-9-228-*t*.)

44. Vincent Cowling and Walter Leighton: *On convergence regions for continued fractions.* (Abstract 49-7-184-*t*.)

45. M. M. Day: *Uniform convexity.* IV. (Abstract 49-9-205-*t*.)

46. John DeCicco: *Kasner's pseudo-angle.* (Abstract 49-9-229-*t*.)

47. Nelson Dunford and D. S. Miller: *On the ergodic theorem.* (Abstract 49-11-257-*t*.)

48. C. J. Everett: *Closure operators and galois theory in lattices.* (Abstract 49-7-180-*t*.)

49. G. M. Ewing: *Existence theorems for multiple integral variation problems.* Preliminary report. (Abstract 49-9-206-*t*.)

50. F. A. Ficken: *Note on the existence of scalar products in normed linear spaces.* (Abstract 49-5-133-*t*.)

51. G. E. Forsythe: *Note on equivalent potential temperature.* (Abstract 49-9-219-*t*.)

52. D. W. Hall: *A note on primitive skew curves.* (Abstract 49-11-307-*t*.)

53. M. H. Heins: *On a problem of Walsh concerning the Hadamard three circles theorem.* (Abstract 49-11-261-*t*.)

54. M. R. Hestenes: *The isoperimetric problem of Bolza in parametric form.* (Abstract 49-11-263-*t*.)

55. F. B. Hildebrand: *On the stress distribution in cantilever beams.* (Abstract 49-9-222-*t*.)

56. Carl Holtom: *Permanent configurations in the n-body problem.* (Abstract 49-11-283-*t*.)

57. C. C. Hsiung: *An invariant of intersection of two surfaces.* (Abstract 49-9-231-*t*.)

58. C. C. Hsiung: *Projective invariants of some particular pairs of space curves.* (Abstract 49-9-232-*t*)

59. C. C. Hsiung: *Theory of intersection of two plane curves.* (Abstract 49-9-233-*t*.)

60. Nathan Jacobson: *An extension of Galois theory to non-separable and non-normal fields.* (Abstract 49-11-245-*t*.)

61. Irving Kaplansky: *Solution of the "problème des ménages."* (Abstract 49-7-181-*t*.)

62. Brockway McMillan: *Random point patterns.* Preliminary report. (Abstract 49-11-265-*t*.)

63. Szolem Mandelbrojt: *Quasi-analyticity and analytic continuation. A general principle.* (Abstract 49-9-208-*t*.)

64. Morris Marden: *On the stream function of axially symmetric flows.* (Abstract 49-9-225-*t*.)

65. A. D. Michal: *An analogue of the Maupertuis-Jacobi "least" action principle for dynamical systems of infinite degrees of freedom.* (Abstract 49-11-288-*t*.)

66. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* I. *Wave motion as a study in geodesics.* (Abstract 49-11-289-*t*.)

67. D. S. Miller: *On Carathéodory and Gillespie linear measure.* (Abstract 49-9-209-*t*.)

68. H. T. Muhly: *Independent integral bases and a characterization of regular surfaces.* (Abstract 49-7-195-*t*.)

69. H. E. Newell: *On the asymptotic forms of the solutions of a linear matric differential equation in the complex domain.* (Abstract 49-11-269-*t*.)

70. Isaac Opatowski: *Cantilever beam of circular cross section and constant strength under the action of its own weight.* (Abstract 49-11-290-*t*.)

71. W. V. Parker: *Limits to the characteristic roots of a matrix.* (Abstract 49-9-202-*t*.)

72. M. O. Peach: *Simplified technique for constructing orthonormal functions.* (Abstract 49-11-291-*t*.)

73. Harry Pollard: *A new criterion for completely monotonic functions.* (Abstract 49-7-190-*t*.)

74. Raphael Salem: *A singularity of the Fourier series of continuous functions.* (Abstract 49-9-211-*t*.)

75. H. E. Salzer: *Table of first two hundred squares expressed as a sum of four tetrahedral numbers.* (Abstract 49-9-203-*t*.)

76. Henry Scheffé: *On a measure-theoretic problem arising in the theory of non-parametric tests.* (Abstract 49-9-239-*t*.)

77. H. M. Schwartz: *Riemann integration with respect to additive interval functions.* Preliminary report. (Abstract 49-11-271-*t*.)

78. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* I. (Abstract 49-9-212-*t*.)

79. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* II. (Abstract 49-9-213-*t*.)

80. Seymour Sherman: *Finite aspect ratio effect in the Glauert-Prandtl regime.* (Abstract 49-9-226-*t*.)

81. W. J. Thron: *A general theorem on convergence regions for continued fractions* $b_0+K(1/b_n)$. (Abstract 49-7-191-*t*.)

82. W. J. Thron: *Convergence regions for the general continued fraction.* (Abstract 49-7-192-*t*.)

83. A. D. Wallace: *A structural property of transformations.* (Abstract 49-9-241-*t*.)

84. F. T. Wang: *On Riesz summability of Fourier series.* III. (Abstract 49-9-217-*t*.)

85. Alexander Weinstein: *On the bending of a clamped plate.* (Abstract 49-9-227-*t*.)

86. Paul White: *On r-regular convergence.* (Abstract 49-9-242-*t*.)

87. Hassler Whitney: *On the extension of differentiable functions.* (Abstract 49-7-193-*t*.)

88. D. V. Widder: *Positive temperatures on an infinite rod.* (Abstract 49-11-295-*t*.)

T. R. HOLLCROFT,
*Associate Secretary*

*Development of the sciences.* Second series. By Oystein Ore, Frank Schlesinger, Henry Margenau, John Arrend Timm, Chester Ray Longwell, Lorande Loss Woodruff, Walter Richard Miles, John Farquhar Fulton. New Haven, Yale University Press; London, Humphrey Milford and Oxford University Press, 1941. 10+336 pp. $3.00.

Keyed to the present interest in scientific advance, these chapters form a short introduction to science, its methods and ideas, background and trends. The presentation is not technical but designed for the general reader. The material was first offered during the winter and spring of 1940 as a series of public lectures sponsored by the Yale University chapter of Gamma Alpha. It comprises discussions by eight well known Yale scientists, representing the fields of mathematics, astronomy, chemistry, physics, geology, biology, psychology, and medicine. Each of the first seven lectures traces the development of one of the basic sciences from its beginnings to its most recent results. The last lecture shows the interdependence of these various sciences as illustrated by specific examples in the history of medicine.

The preceding statements are quoted literally from the jacket of the book under review. Since the book comprises less than three hundred and fifty pages, it is obvious that the presentation is extremely concise. But the reviewer found that it makes interesting and stimulating reading, and feels that the book cannot fail to convey to the general reader an overwhelming impression of the power of the human mind engaged in unselfish and purposeful endeavors. The chapter on mathematics, written by O. Ore, comes up in every way to the well known standards of depth of thought and clarity of presentation of its author. This chapter should be very valuable to graduate students of mathematics who, as a rule, have little help in obtaining an integrated picture of mathematics from the historical point of view. It is to be hoped that this chapter may be made available to this particular group of readers in some mathematical journal, possibly in somewhat extended form.

TIBOR RADÓ

*The methodology of Pierre Duhem.* By Armand Lowinger. New York, Columbia University Press, 1941. 184 pp. $2.25.

Pierre Duhem (1861–1916) was an outstanding French physicist whose work was mainly concerned with thermodynamics, but he

was one of those scientists who not only walk on scientific paths but want to understand the history of discoveries and the different methods used in scientific proceedings.

The greater part of the present book is a well ordered report on the writings of Duhem on methods in theoretical physics. These investigations have two aspects. On the one hand, Duhem tries to describe objectively the actual methods; on the other hand, he cannot suppress his temperament and speaks, appreciatingly or blamingly, about the different methods and gives prescriptions about the aims and the proceedings. He belongs to that group of physicists of the latter half of the last century who could be called methodical ascetics. They asserted that the theoretical physicist has only the task to *describe* and order the physical phenomena but not to *explain* them. They tried to suppress the natural demand to find out *causes* for the phenomena, a demand which requires the connecting of the phenomena through intermediate tensions and phases which are not directly observed. A more detailed analysis would show that it is impossible to avoid in the description such intermediate, invisible, "theoretical" phases of the pair, cause and effect.

Another object of dislike for Duhem are the "models" in physics, for instance the electrical models of the English school of physicists. He admits that the models sometimes suggest valuable ways of progress but he does not like them. They are not abstract enough, too near to the tempting but forbidden field of "explanation." He would, probably, have been astonished to see the progress of modern molecular physics made with the help of the atom model. From the logical point of view there is no precise distinction between representation through mathematical symbols and through models.

Duhem is somewhat subjective in his views and indulges in dangerous generalizations and summaries in describing the different kinds of minds. There is the broad but feeble mind (mainly English) which uses models, and the narrow but strong mind using the method of abstract reasoning (French and German). Later, after the first world war had started, he discovers also some characteristics of the "German mind." It is "geometric"; it treats all things by the axiomatic method and gives birth to theories which deny obvious truth, for instance, the various non-Euclidian geometries. This assertion is a good example of the innate tendency of the human understanding to order and simplify the wealth of phenomena, outside physics as well as in it, and of the manner in which prejudice, antipathy, and sympathy help to achieve this order.

The last chapter gives critical remarks and conclusions of the

author. Some of the remarks are given above. I do not quite share his opinion that the "methodologist" has only to describe the methods used by the physicists or the scientists in general. Even a non-expert may see sometimes, by general considerations, ways which the researchman should go. Plato and Aristotle were no mathematicians, Bacon no physicist in a proper sense, but, undoubtedly, they furthered the development of mathematics and physics. That Aristotle was an impediment to the development of physics was the fault of "experts" who adhered in a slavish way to his physical theories. The outsider *sometimes* sees more of the general landscape of science, where the scientific workers go the toilsome ways which lead to discoveries.

In his conclusions the author presents certain "indefeasible facts" of the scientific situation, for instance that in modern physics metaphysical explanatory theories are excluded. His last "conclusion" is somewhat metaphysical and not easily accepted, namely that science approaches "asymptotically" a perfectly adequate account of reality. And it is given as an "indefeasible fact of the human situation—that the Human Spirit is one and that the different activities in which it expresses itself must in the end arrive at the same conclusion." It is difficult for me to imagine common conclusions which will be reached "asymptotically" by the Human Spirit in its metaphysical, social and scientific activities.

Only one little critical remark: the mathematical reader will be astonished to find on page 94, quoted from Poincaré, an erroneous description of the method of mathematical induction.

M. DEHN

*Transients in linear systems, studied by the Laplace transform.* Vol. I. By M. F. Gardner and J. L. Barnes, New York, Wiley; London, Chapman and Hall, 1942. 9+389 pp. $5.00.

This book is remarkable in that it is perhaps the first serious attempt to present the theory of the Laplace transform to the mathematics or engineering student at an early stage of his studies. For the complete understanding of this transform it is necessary to have mastered the theory of functions of a complex variable. This fact has hitherto barred many a student from the use of a valuable analytic tool until rather late in his mathematical career. The present work shows conclusively that this delay is unnecessary.

The authors are able to place the fundamental idea of the method before the reader even in the first chapter. The essence of the matter consists in replacing by the Laplace transform one function space

by another in such a way that differential and integral operations on the functions of the first space become algebraic operations on the functions of the second. Then an integrodifferential equation becomes an algebraic equation. Let us illustrate by finding that solution of the differential equation $dy/dx-y=x$ for which $y(0)=0$. One first defines the Laplace transform $Y(s)$ of the required solution $y(x)$:

$$Y(s) = \int_0^\infty e^{-sx}y(x)dx.$$

Integration by parts shows that

$$\int_0^\infty e^{-sx}y'(x)dx = -y(0) + sY(s),$$

so that the given differential equation transforms into

$$sY(s) - y(0) - Y(s) = s^{-2}.$$

After introducing the boundary condition $y(0)=0$ one easily solves the algebraic equation and finds that

$$Y(s) = (s-1)^{-1} - s^{-1} - s^{-2}.$$

It now only remains to find the function $y(x)$ which has this function for its transform. This is easily read off from a table of transforms and is found to be $y(x)=e^x-x-1$.

A critical examination of the above solution will show that certain assumptions have been made, notably that the unknown function $y(x)$ and its derivative have Laplace transforms. The authors take the perfectly acceptable point of view that such assumptions are justified by the results. It is always possible to check directly the solution of a differential system. If it is correct, no further investigation of the method is necessary. If the brilliant success arouses the curiosity of the student, so much the better. He can learn why it all works when he is better equipped with mathematical tools. One outstanding advantage of the method is brought out in the above example. Due to the fact that the transform is unilateral (the integration runs from 0 to $+\infty$ and not from $-\infty$ to $+\infty$), one-point boundary conditions are brought automatically into the solution. Thus no general solution of the differential equation with arbitrary constants is involved.

There is a very welcome second chapter in which the physical background of the electrical and mechanical problems treated is discussed in great detail. Most comparable texts assume that the stu-

dent is already familiar with the physics of the situation, and if he is not, he must look elsewhere for help. This chapter is independent of the Laplace method and could be used in connection with any course where practice in setting up differential systems from physical problems is required.

It is natural that the authors should wish, in a book of this character, to keep the proof of theorems a secondary matter. Thus in chapter 3 the Laplace transform is introduced by analogy with the more familiar Fourier series and integrals. Even in chapters 4 and 5, where a more formal approach is undertaken, demonstrations of theorems are either omitted entirely (Theorems 1, 2, 3 and 4) or sketched briefly. This procedure seems not to detract from the essential clarity of the presentation. The remainder of the book, with the exception of chapter 8, elaborates further and illustrates profusely the method of solving integrodifferential and difference equations.

The reviewer's chief adverse criticism of the book has to do with chapter 8. Here the authors seem to abandon the general plan of the book. Some twenty theorems of a more advanced nature about the Laplace transform are developed. Demonstrations are given. No use of the results is made in the present volume. In several cases the theorems are incorrectly stated, in others the proofs are wrong or at least inadequate. One of the more important errors stems from indiscriminate interchange of limit and integral operations (though elsewhere the authors are careful about this). Another is based on the incorrect assumption that a Laplace integral defines an analytic function which must have a singularity on the axis of convergence. No doubt the chapter was included to prepare the way for a second volume on partial differential equations and related material which is to follow. The flaws in the chapter need not impair essentially the usefulness of the present volume. For, the student who is looking merely for technique may omit the chapter, the brilliant student may read it with circumspection. It may be worth while to call attention to one other misstatement in chapter 4. Here it is stated that a function of a complex variable which has a first derivative at a point necessarily has all higher derivatives there also (compare the function $|z|^2$).

In conclusion it may be well to summarize the special features of the book. First and foremost, it places a vital tool well within the grasp of a student with meager mathematical training. It dispells the mystery of the Heaviside operational calculus by showing the real basis of that technique. It gives complete physical background for the

integrodifferential and difference equations it solves. It has a wealth of illustrative examples, done in the text, and many problems for the student at the ends of the chapters (no answers). There is a very extensive table of Laplace transforms, as useful as a table of integrals in a calculus course. In chapter 1 and in appendix B a valuable comparison of the Laplace method with other possible techniques is given. Historical notes on the mathematical theory appear in appendix C. Finally there is one of the most extensive bibliographies on the subject yet to appear.

D. V. WIDDER

*A treatise on projective differential geometry.* By Ernest Preston Lane. Chicago, University of Chicago Press, 1942. 9+466 pp. $6.00.

Since the appearance of the author's earlier volume, *Projective differential geometry of curves and surfaces* (University of Chicago Press, Chicago, 1932), significant contributions to the field of projective differential geometry have been made by geometers in various parts of the world. In the preface the author does not claim that the present treatise is exhaustive, but states that it represents the fruit of ten years of study and investigation and gives an account of the author's experience with those portions of the subject which interested him most. It is with respect to those portions of the subject, then, that the reviewer interprets the author's statement, appearing earlier in the paragraph, that "the present volume integrates the new material with the old and gives a connected exposition of the theory to date." The treatise reports results of studies made by many workers in the field, and expounds a wide range of topics. It is also notable, however, that some of the newer topics that have attracted rather general interest have not been mentioned. Parts of the proofs, involving calculations that are difficult for the uninitiated, are so frequently left to the care of the reader that only the more advanced students of the subject can use the volume successfully as a textbook. The treatise is properly designed to serve as a reference book for the research worker in the field. Geometric concepts and results are consistently described in a lucid graphic manner.

It is satisfying to observe that the present volume devotes considerably more attention to the methods of Wilczynski on the study of curves by means of linear differential equations than does the earlier volume. It is disappointing, however, to find no mention made of Stouffer's simplifications of Wilczynski's methods of determining (i) canonical power series expansions for the local equations of plane and space curves, and (ii) the geometric characteriza-

tions of the vertices of the associated reference triangles and tetrahedra. About three times as much space is afforded the theories of curves as was allotted to this subject in the earlier volume. A similarly expanded amount of space is afforded the subject of surfaces in the present volume. The metric and affine applications are omitted. The Fubini method of the use of differential forms is omitted. The theorems stated in the earlier volume as exercises appear without proofs at the ends of the sections as supplementary theorems. Topics in the study of conjugate nets are treated much more fully than in the former volume. In particular, three types of sequences of Laplace are treated extensively in the present volume, whereas the subject of sequences of Laplace received scant attention in the earlier volume. A twenty page chapter is devoted to the study of plane nets. Grove's studies of the neighborhoods of singular points of surfaces are not mentioned. Their omission represents a distinct loss. A brief treatment of Fubini's theory of W-congruences is given. This theory could have been brought more nearly up to date had the contributions of Jonas (1937) and Fubini (1940) been added.

The subjects of plane and space curves in hyper- and ordinary space are treated in the first three chapters. The next three chapters deal with surfaces in ordinary projective space. Two chapters are devoted to the study of conjugate nets and one is devoted to the theory of plane nets. Transformations of surfaces and surfaces and varieties in hyperspace are studied in the last two chapters. The exposition is uniformly excellent throughout the volume. No attempt will be made to review each chapter by topics, but to give an indication of the nature of the relationship of the material included in the treatise to other material in the literature a topical review is here given of the subject matter contained in the two chapters on surfaces in ordinary space.

Chapters V and VI contain expositions of fundamental topics in the theory of analytic surfaces in ordinary three-dimensional space. The parametric curves on each surface under consideration have been selected to be the asymptotic curves. The form of the fundamental system of defining differential equations is deduced from a consideration of the curvilinear differential equation of the asymptotic curves of a surface. Integrability conditions are discussed and a study is made of the effects on the differential equations of the most general transformations of variables which leave the surface undisturbed and preserve the parametric character of the asymptotic curves. Use is then made of special transformations of proportionality factor, from original homogeneous coordinates $x$ to new coordinates $\bar{x}$, to

obtain reductions of the system of differential equations to canonical forms. The first canonical form results from the choice of coordinates $\bar{x}_i$ defined by $\bar{x}_i = x_i/x_4$. The second and third canonical forms are Wilzynski's and Fubini's, respectively. Both are obtained by making a transformation of the form $x_i = \lambda \bar{x}_i$. For Wilczynski's canonical form $\lambda = e^{\theta/2}$, and for Fubini's canonical form $\lambda = (e^{\theta}/\beta\gamma)^{1/2}$, wherein $\theta$, $\beta$, $\gamma$ are coefficients of the original differential equations of the surface. The canonical forms of Wilczynski and Fubini have been used extensively by various geometers. In order to arrive at Fubini's canonical form it is necessary that the surface under consideration be not ruled. No mention is made in the present volume of Grove's important class of canonical forms. This class not only contains Fubini's canonical form but also contains other canonical forms of similar geometric interest which, in marked contrast with Fubini's canonical form, exist whether or not the surface is ruled. A brief but interesting account is given in the present treatise of a method of calculation of an expansion which represents one nonhomogeneous coordinate $z$ as a power-series in the two nonhomogeneous coordinates $x$ and $y$ of a point $X$ of a surface. A great expenditure of labor is involved in the calculation of the terms of the fifth and sixth degrees of this power-series. The theory and method of procedure are adequately explained; the tedious details of the calculations are appropriately omitted. The author has contributed much to the development of such power-series expansions. A canonical form for the differential equations of a ruled surface is obtained in a direct and simple manner. The relations between the coefficients of these equations and those on which Wilczynski based his theory of ruled surfaces are then established. At this point properties of the curves of sections by planes through an asymptotic tangent are deduced. The flecnode curves are briefly considered. The quadrics of Darboux are defined and their equations with reference to a local coordinate system are obtained. The quadric of Lie and the asymptotic osculating quadrics of Bompiani are also defined and their local equations are calculated. Bompiani's characterization of the quadric of Wilczynski as a special asymptotic osculating quadric is deduced. A somewhat similar characterization of the quadric of Fubini is reached. Reciprocal congruences are defined and studied, and their developables and focal surfaces are determined. Although G. M. Green pioneered in this subject of reciprocal congruences by his study of congruences in the relation $R$, no reference is made to Green's work. The author's presentation appears to be essentially an interpretation of Green's work in terms of a different system of differen-

tial equations than was employed by Green. The equations and expressions are simpler in the forms in which they were originally expressed by Green, in terms of the coefficients of Wilczynski's canonical form. Considerable attention is given to the geometric characterization of various canonical lines, but no reference is made to a known unified geometric characterization of the canonical pencil.

Chapter VI opens with the introduction of the point-to-plane correspondence in which each point of a surface corresponds to the tangent plane to the surface at the point. This correspondence has been thoroughly studied by the author and Fubini and Čech. The axis, ray, associate axis, and associate ray are defined at a point in connection with a conjugate net, and relations among these are established. The author's studies of a general pencil of conjugate nets and the associated ray-point cubic and ray-conic are presented. The cones which correspond to the ray-point cubic and the ray-conic by the point-to-plane correspondence mentioned above are determined. These are the axis plane cone and the axis quadric cone, respectively. Hypergeodesics are defined on a surface by means of a curvilinear differential equation of the second order. Several properties of hypergeodesics are enumerated. A cone of the third class is determined as the envelope at a point of the surface of all of the osculating planes of the hypergeodesics of a family passing through that point. The cone has three cusp planes which intersect in a line called the cusp-axis of the system of hypergeodesics at the point. It is shown that the curves of a pencil of conjugate nets on a surface form a family of hypergeodesics. Other families of hypergeodesics are discussed, among which are the projective geodesics and the union curves of a congruence $\Gamma_1$ on a surface. The projective geodesics are the extremal curves of the invariant integral

$$\int_{u_0}^{u} (\beta\gamma v')^{1/2} du$$

known as the projective arc length of the curve along which it is calculated.

The property of the projective normal which Fubini discovered is established, namely, at a point on a surface the cusp-axis of the projective geodesics is the projective normal. The quadric of Moutard for a direction at a point of a surface is defined and its local equation is calculated. The author describes some properties of the transformations of Čech and presents his own construction for the point $P_k$ which corresponds to a given plane in the general transformation of

Čech at a point $P_x$ of the surface $S$. The author fails to mention the generalization of this transformation by Fubini and Čech. A simple construction is known for the elements which correspond in this more general transformation. The pangeodesics are defined as the extremal curves of the integral invariant

$$\int_{u_0}^{u} \left(\frac{\beta + \gamma v'^3}{v'}\right) du.$$

The geometric characterizations of the pangeodesics obtained by Lane and Segre appear in the volume. Formulas for the differentiation of local point coordinates are deduced and it is proved that with the use of these formulas the ordinary theory of envelopes may be applied in finding the envelope of a family of surfaces $S_1$ defined by $f(x_1, x_2, x_3, x_4, u, v)=0$ in which $x_1, \cdots, x_4$ are the local coordinates of a variable point on the surface $S_1$ and $u$, $v$ are the curvilinear coordinates of the point $x$ on the surface $S$. An application of this theory of envelopes is given in the determination of the local coordinates of the vertices of the tetrahedron of Demoulin. Results of L. Green, B. Segre, Su, and Thomsen and Mayer, on the respective subjects, the envelope of the quadric of Moutard, projective curves, projectively minimal curves, and projectively minimal surfaces, are reported as supplementary theorems. Important concepts which might have been included are those of projective curvatures and torsions of a curve on a surface, with respect to differential forms of the surface. New characterizations of the curves of Darboux, the curves of Segre, and of the Darboux-Segre pencil of conjugate nets could have been given. Also simple geometric characterizations of the important integral invariants

$$\int_{u_0}^{u} (\beta\gamma v')^{1/2} du, \qquad \int_{u_0}^{u} \left(\frac{\beta + \gamma v'^3}{v'}\right) du,$$

are known, but none are given.

The author has on former occasions called the attention of differential geometers to the need of an intrinsic tensor calculus for the study of a variety $V_k$ immersed in a projective space $S_n$, where $n>k>1$ and $k\neq n-1$. In the present treatise he has neglected to announce that Grove has recently developed such a tensor calculus. Grove has shown that certain tensors arising in the theory of the geometry of paths on a variety $V_k$ can be expressed in terms of tensors arising in the study of the variety from the point of view of classical projective differential geometry. The works of E. Cartan

on projective differential geometry have also been overlooked. Cartan has used his method of moving reference systems with outstanding effectiveness to study both classical projective differential geometry and the generalized projective differential geometry of spaces with a projective connection.

To form a fair estimate of the treatise, in view of numerous omissions of important topics, we must keep in mind that the aim of the author was not to write an exhaustive treatise on the subject but to present an account of his experience with those portions of the subject which interested him most.

The style of the book is attractive, the typography is clear and restful to the eye, and the proofreading has been carefully done. The research worker will find the volume informative, suggestive, and stimulating.

P. O. Bell

## NOTES

The Indian Mathematical Society will hold its next conference at Annamalai University at Annamalainagar about the end of December, 1943. In connection with the conference, it is proposed to have a mathematical exhibition intended to illustrate the richness and variety of the subject and the wide range of its applicability to life situations.

Miss Louise Adams has been appointed to an assistant professorship at High Point College, High Point, North Carolina.

Assistant Professor C. K. Alexander of Occidental College, Los Angeles, California, has been promoted to an associate professorship.

Associate Professor H. F. Archibald of Keuka College, Keuka Park, New York, has been promoted to a professorship.

Dr. H. G. Ayre of Western Illinois State Teachers College has been promoted to an associate professorship.

Assistant Professor H. M. Bacon of Stanford University has been promoted to an associate professorship.

Dr. G. A. Baker of the University of California at Davis has been promoted to an assistant professorship.

Associate Professor A. C. Berry of Lawrence College, Appleton, Wisconsin, has been promoted to a professorship.

Assistant Professor J. W. Blincoe of the University of Tennessee has been appointed to a professorship at Randolph-Macon College.

Dr. J. O. Blumberg of the University of Pittsburgh has been promoted to an assistant professorship.

Mr. W. H. Bradford of the John McNeese Junior College of the Louisiana State University has been promoted to an assistant professorship.

Mr. R. L. Calvert of the Utah State Agricultural College has been promoted to an assistant professorship.

Assistant Professor C. E. Clark of Bradley Polytechnic Institute has been appointed to an associate professorship at Emory University.

Dr. J. B. Coleman has been appointed to a professorship at the University of Richmond.

Dr. M. W. Dehn of Illinois Institute of Technology has been appointed to a professorship at St. John's College, Annapolis, Maryland.

Dr. R. W. Gibson of the University of Illinois has been appointed to an assistant professorship at Kansas State College.

Dr. Edward Helly of Monmouth Junior College has been appointed visiting lecturer at the Illinois Institute of Technology.

Dr. Fritz Herzog of Cornell University has been appointed to an assistant professorship at Michigan State College.

Mrs. Phyllis H. Hutchings has been appointed assistant professor of astronomy and mathematics at Whitman College, Walla Walla, Washington.

Associate Professor Nathan Jacobson of the University of North Carolina has been appointed to an associate professorship at Johns Hopkins University.

Dr. S. A. Jennings of the University of British Columbia has been promoted to an assistant professorship.

Dr. Lois Kiefer of the University of Illinois has been appointed head of the department of mathematics at State Teachers College, Silver City, New Mexico.

Professor M. G. Moore of Tri-State College, Angola, Indiana, has been appointed to an assistant professorship at Bradley Polytechnic Institute.

Dr. Haim Reingold of the Illinois Institute of Technology has been promoted to an assistant professorship.

Assistant Professor M. A. Sadowsky of the Illinois Institute of Technology has been promoted to an associate professorship.

Dr. Robert Schatten has been appointed to an assistant professorship at the University of Vermont.

Dr. C. L. Seebeck of the University of North Carolina has been appointed to an assistant professorship at the University of Alabama.

Dr. W. S. Snyder of Illinois Institute of Technology has been appointed to an assistant professorship at the University of Tennessee.

Dr. C. J. Thorne of Washington University has been appointed to an assistant professorship at Louisiana State University.

Assistant Professor T. L. Wade of the University of Alabama has been appointed head of the department of mathematics at Florida State College for Women.

Assistant Professor L. R. Wilcox of the Illinois Institute of Technology has been promoted to an associate professorship.

The following appointments to instructorships are announced: Bard College, Columbia University: Dr. H. B. Mann; Harvard University: Mr. R. E. Lane; Illinois Institute of Technology: Mr. Furio Alberti, Dr. Ruth Ballard, Mr. P. L. Browne, Mr. H. J. Miser; Indiana University: Dr. Marion D. Wetzel; Massachusetts Institute of Technology: Mr. G. N. Raney; Michigan State College: Dr. B. H. Bissinger; University of Minnesota: Dr. Louis Garfin; University of Pennsylvania: Mr. R. T. Luginbuhl; University of Rochester: Dr. H. S. Kieval; Syracuse University: Mr. F. L. Celauro, Mrs. Ruth E.

Goodman; Vassar College: Miss Janet C. Durand; Whitman College: Mrs. Pearl C. Miller.

Mr. J. K. Gore, president of the Prudential Insurance Company since 1934, died June 22, 1943, at the age of 79 years.

President C. C. Jones of the University of New Brunswick died August 19, 1943, at the age of 71 years.

Mr. L. L. Locke who retired in 1942 died August 28, 1943. He had been a member of the Society since 1900.

Mr. T. C. Rafferty, consulting actuary, Missouri State Life Insurance Company, died September 10, 1943. He had been a member of the Society since 1925.

Mr. H. M. Showman, lecturer and registrar of the University of California at Los Angeles, died June 24, 1943.

Professor Emeritus E. B. Van Vleck of the University of Wisconsin died on June 2, 1943. He had been a member of the Society since 1893.

Mr. A. W. Whitney died on July 27, 1943. He had been a member of the Society since 1902.

## ABSTRACTS OF PAPERS

### SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

### ALGEBRA AND THEORY OF NUMBERS

243. Angeline J. Brandt: *The free Lie ring and Lie representations of the full linear group.*

The present paper is a continuation and amplification of a paper of Thrall (Amer. J. Math. vol. 64 (1942) pp. 371–388). In the latter paper a recursion formula was developed from which the irreducible constituents of the $m$th Lie representation for $m \leqq 10$ were obtained. The main result of the present paper is a direct formula for the character of the $m$th Lie representation, namely $[m] = [\sum \mu(d) s_A^{d'}]/m$ where $dd' = m$, the sum is over all divisors $d$ of $m$, $\mu(d)$ is the familiar Möbius function and $s_A$ is the trace of $A^d$, $A$ being an arbitrary element of the full linear group. The nature of the irreducible invariant subspaces of $L^m$ (all forms of degree $m$ in the free Lie ring) is determined by this formula for $m \leqq 14$. Certain necessary and certain sufficient conditions are determined in order that a given ideal $I$ be characteristic. Since the intersection of $I$ with the module $L^m$ is an invariant subspace of $L^m$ relative to the representation of the full linear group afforded by $L^m$, it becomes important to know what elements constitute such a subspace and a method for determining this is developed. (Received October 1, 1943.)

244. R. H. Bruck: *Some results in the theory of quasigroups.*

This paper is primarily intended as an illustration of the usefulness of isotopy in quasigroup theory and as ground work for a later paper on linear non-associative algebras. It is largely devoted to the theory and construction of quasigroups with the inverse property. A quasigroup $Q$, finite or infinite, has the inverse property if there exist two one-to-one reversible mappings $L$, $R$, not necessarily distinct, of $Q$ on itself, such that $b^L \cdot ba = ab \cdot b^R = a$ for all $a$, $b$ of $Q$. Also contained in the paper are several new theorems on Moufang quasigroups, an explicit construction of all (Murdoch) abelian quasigroups, and necessary and sufficient conditions that the direct product of two finite quasigroups should contain no sub-quasigroup except itself. (Received August 13, 1943.)

245. Nathan Jacobson: *An extension of Galois theory to non-separable and non-normal fields.*

The extension of Galois theory obtained in this paper is based on two important concepts: self-representation of a field P, that is, a representation of P by matrices with elements in P, and composite (ring) of P with itself. The latter is a generalization of the usual concept of a composite field. With any two self-representations, associate

a product self-representation obtained by substituting for the elements of the matrices of the first representation the matrices that represent these elements in the second representation. This leads to a definition of the product of composites. The latter concept is used to define Galois composites. The fundamental theorem then establishes a (1-1) correspondence between the Galois composites of P and the subfields $\Phi$ of P over which P is finite. This result gives a theorem recently announced by Kalujnine and specializes still further to the classical theorem for subfields $\Phi$ over which P is finite, separable and normal. In addition to the main result a general study of self-representations of fields is given in this paper. (Received August 9, 1943.)

246. R. E. Johnson: *On the equation $\chi\alpha=\gamma\chi+\beta$ over an algebraic division ring.*

The main purpose of this paper is to give necessary and sufficient conditions in order that the equation $\chi\alpha=\gamma\chi+\beta$ have a solution over an algebraic division ring. If $\alpha$ and $\gamma$ are not transforms of each other, this equation always has a unique solution. In case $\gamma$ equals $\alpha$, this equation has a solution if and only if $\beta\alpha\beta^{-1}$ is a unilateral solution of a particular equation over the division ring. For the special case in which the division ring is a quaternion algebra, $\chi\alpha=\alpha\chi+\beta$ has a solution if and only if $\beta\alpha=\bar{\alpha}\beta$, where $\bar{\alpha}$ is the quaternion conjugate to $\alpha$. (Received September 20, 1943.)

247. R. E. O'Connor and Gordon Pall: *On integral quadratic forms in many variables.*

A difficult problem proposed by Ko and Erdös, to find an integral quadratic form of determinant 1 and minimum greater than 2, is solved by the construction of a form in 24 variables of minimum 3, and a similar form in 40 variables. Improvements are found on the limits within which quadratic forms in genera of one class may exist. (Received August 6, 1943.)

248. R. R. Stoll: *Representations of completely simple semigroups.*

Every finite semigroup (that is, a finite system closed under an associative multiplication) can be represented by a set of correspondences of a suitable finite set to itself. This suggests the problem of the determination of all such representations of a given semigroup. Towards this end semigroups of correspondences are first studied. It is shown that if $P$ is a semigroup on the set $N$ then $N$ can be written as the class sum of mutually disjoint subsets which form "weakly transitive" systems for $P$. Next, each weakly transitive system has a unique decomposition as the union of "transitive" and "quasi-transitive" systems. This reduces the representation problem to the determination of all transitive and quasi-transitive representations and methods for composing such representations. The above results are used to determine all representations of a type of finite semigroup called a "Kerngruppe" by Suschkewitsch (Communications de la Société Mathématique de Kharkow vol. 6 (1933) pp. 27–38) and a "completely simple semigroup without zero" by Rees (Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400). (Received October 1, 1943.)

249. Dorothy M. Stone: *An algebraic characterization of measure algebras.* Preliminary report.

Let $E$ be a non-atomic Boolean $\sigma$-algebra. This paper gives purely algebraic conditions on $E$ which are necessary and sufficient for the existence of a countably addi-

tive, real-valued, non-negative, finite measure function on $E$, vanishing only for $\phi$, the zero of $E$. $E$ is given the star-topology of G. Birkhoff, and it is shown that $E$ will have such a measure if and only if it is metrizable in a certain way in this topology. It is next shown that $E$ is a topological space if and only if it satisfies a certain distributive law. A basis for the neighborhoods of $\phi$ can then be characterized algebraically, making it possible to state simple algebraic equivalents for the various separation axioms. If $E$ satisfies the countable chain condition, $T_3$ implies metrizability, which gives an outer measure on $E$. A measure is obtained by means of an additional algebraic requirement. Thus, if $E$ satisfies the distributive law referred to, the countable chain condition, the algebraic equivalent of $T_3$, and the additional requirement, there exists a measure-function on $E$. These conditions are easily seen to be necessary. It is not known whether they are independent. (Received October 2, 1943.)

## 250. R. M. Thrall: *On the decomposition of modular tensors.* II.

Let $G$ be the $n$-rowed full linear group over a field $k$ of characteristic $p$. A representation of $G$ is called a tensor representation if its space is a direct sum of subspaces and factor spaces of tensor spaces. A main result of the present paper is that for a finite field $k$, the $k$-group ring of $G$ has a faithful tensor representation. In paper I the representations afforded by all tensors of rank $m<2p$ were determined subject to the condition that $k$ has more than $p$ elements. In this paper the same is done for the field $k$ with $p$ elements. A main tool in this investigation is the construction of a representaton of $G$ from each irreducible representation of the non-modular full linear group, and a corresponding extension of the Brauer-Nesbitt modular character theory to this case. The presence of zero divisors in the ring of polynomial functions over a finite field enters into the treatment of the case of tensors of rank $2p-1$ over a two-dimensional vector space, and the situation in that case should help point the way to the general decomposition theory. (Received October 1, 1943.)

## Analysis

## 251. Stefan Bergman: *Fundamental solutions of partial differential equations of the second order.*

As was previously shown, for every differential equation $L(U)=U_{z\bar{z}}+H(Z,\bar{Z})U=0$, $Z=X+iY$, $\bar{Z}=X-iY$, there exists a function $\mathrm{E}(Z,\bar{Z},t)=1+Z\bar{Z}t^2\mathrm{E}^*(Z,\bar{Z},t)$ such that $U=\mathrm{P}(f)\equiv\int_{-1}^{+1}\mathrm{E}(Z,\bar{Z},t)f(Z(1-t^2)/2)dt/(1-t^2)^{1/2}$, where $f$ is an arbitrary analytic function, is a solution of $L(U)=0$ (see Duke Math. J. vol. 6 (1940) p. 537). The author shows that a fundamental solution $\Gamma(z,\bar{z},\zeta,\bar{\zeta})$ of the equation $S(v)=v_{z\bar{z}}+F(z,\bar{z})v=0$ is given by $\mathrm{P}(1/2\pi)\log|Z|+G(Z,\bar{Z})$. Here $Z=z-\zeta$, $\bar{Z}=\bar{z}-\zeta$, and P is the operator introduced above for the equation $L(U)=0$ with $H(Z,\bar{Z})=F(Z+\bar{\zeta},\bar{Z}+\bar{\zeta})$. $G(Z,\bar{Z})=-\int_0^{\bar{z}}\int_0^{z}D(Z,\bar{Z})dZd\bar{Z}+\int_0^{\bar{z}}\int_0^{z}H(Z,\bar{Z})(\int_0^{\bar{z}}\int_0^{z}D(Z,\bar{Z})dZd\bar{Z})dZd\bar{Z}+\cdots$ where $D(Z,\bar{Z})=(1/2)\int_{-1}^{+1}t^2[2\mathrm{E}^*+\bar{Z}\mathrm{E}^*_{\bar{Z}}+Z\mathrm{E}^*_{Z}]dt/(1-t^2)^{1/2}$. Using the representations of functions $v(z,\bar{z})$, $S(v)=0$, in the form of a line integral over a closed curve in terms of $\Gamma$, $\partial\Gamma/\partial n$, $v$ and $\partial v/\partial n$ the author studies the growth of $v$ and $\partial v/\partial n$ along circles $|z|=r$, $r\to\infty$. The existence of an analogous function $\mathrm{E}(X,Y,t)$ for every equation $H(U)=U_{XY}+H(X,Y)U=0$ (of hyperbolic type) has been established (see above reference). $(1/2\pi)\int_{-1}^{+1}\mathrm{E}(X,Y,t)dt/(1-t^2)^{1/2}$ is now shown to be the Riemann function of the equation $v_{xy}+F(x,y)v=0$, where $X=x-\xi$, $Y=y-\eta$ and $H(X,Y)=F(X+\xi,Y+\eta)$. Analogous relations hold for more general equations. (Received September 11, 1943.)

252. B. H. Bissinger: *Generalizations of continued fractions.*

The simple continued fraction for a real number $x$, $0<x<1$, may be written in the form $f(a_1+f(a_2+\cdots$, where the $a$'s are the partial denominators and $f(t)=1/t$. The present paper generalizes the simple continued fractions by using functions $f(t)$, other than $1/t$. When $f(t)$ belongs to an appropriate class $F$ of real functions, defined for $t\geqq 1$, and $a_1, a_2, \cdots$ is a sequence of positive integers, the "$f$-expansion" $f(a_1+f(a_2+\cdots$ converges to a number $x$, $0<x<1$. The "$f$-expansion of $x$," $0<x<1$, is given by an algorithm equivalent to the euclidean algorithm when $f(t)=1/t$. The main result of the paper is the identity of the $f$-expansion of $x$ with the $f$-expansion which converges to $x$. The remainder of the paper is concerned with generalizations of certain results in the theory of simple continued fractions, established by Borel and F. Bernstein. One of these results is that, for almost all $x$, $0<x<1$, the $a$'s form an unbounded sequence. This and theorems of a similar nature are proved for generalized continued fractions, provided that $f(t)$ belongs to a certain subclass of $F$. (Received August 5, 1943.)

253. Fritz Herzog and B. H. Bissinger: *Generalization of Borel's and F. Bernstein's theorems on continued fractions.*

Let $f(a_1+f(a_2+\cdots$ be the $f$-expansion of $x$, $0<x<1$, as defined by Bissinger in his paper *Generalizations of continued fractions* (abstract 49-11-252). For the case $f(t)=1/t$ (simple continued fractions) Borel and F. Bernstein have proved: (a) the set of $x$, $0<x<1$, for which the $a$'s are bounded has measure zero; (b) the set of $x$, $0<x<1$, for which all $a$'s are greater than unity has measure zero. In the paper referred to above the statements (a) and (b) were proved to hold when $f(t)$ belongs to a certain class of polygonal functions. The present paper is concerned with the problem of characterizing analytically a class of functions which does include $f(t)=1/t$ and such that statements (a) and (b) hold when $f(t)$ belongs to that class. This investigation at the same time reveals by which analytic properties of the function $f(t)=1/t$ the original statements of Borel and F. Bernstein may be proved. (August 5, 1943.)

254. R. C. Buck: *A note on subsequences.*

Starting from a result of H. Steinhaus (Prace Matematyczno-Fizyczne vol. 22 (1911) p. 129), it is shown that a sequence $s_n$, limitable by a regular matrix method $T$, is convergent if and only if each of its subsequences is also limitable $T$. This general tauberian theorem gives rise to a similar result for series. If there exists a regular matrix method $T$ which sums every series obtained from $\sum a_n$ by bracketing blocks of terms, then $\sum a_n$ is itself convergent. (Received September 11, 1943.)

255. R. C. Buck: *Multiple sequences.*

An $r$-multiple sequence of points of a limit space $L$ is a single value function $p(j)$ from $\mathfrak{J}$ to $L$, where $\mathfrak{J}$ is the $r$-fold cartesian product of the integers with themselves. Under an inclusion ordering, $\mathfrak{J}$ is a directed system; a subsequence may then be defined in terms of a monotonic function $\lambda(j)$, defined on $\mathfrak{J}$ to $\mathfrak{J}$. In terms of a product measure (C. Visser, Studia Mathematica vol. 7, p. 143) it is shown that the set of all convergent subsequences of $p(j)$ form a measurable set, with measure 0 or 1. By means of a lemma on measure preserving translations and sets of measure 1, it is shown that if $p(j)$ is divergent, so are almost all of its subsequences. This is a generalization of a previous result of H. Pollard and the author. (Received September 29, 1943.)

256. R. C. Buck and Harry Pollard: *Convergence and summability properties of subsequences.*

The authors discuss the relation of the convergence or summability of a sequence to that of its subsequences. The set of subsequences of a sequence can be mapped in a 1:1 fashion on the interval (0, 1). In terms of Lebesgue measure on this set, it is shown that a sequence is convergent if and only if almost all of its subsequences are convergent. Similarly, for bounded sequences, a sequence is $(C, 1)$ summable if and only if almost all of its subsequences are $(C, 1)$ summable. For unbounded sequences, $(C, 1)$ summability of $s_n$ follows from that of almost all of its subsequences, but not conversely. The principal tools are the properties of homogeneous sets, and the Rademacher functions. These results are intimately connected with probability; in particular, a theorem of Birnbaum and Zuckerman (Amer. J. Math. vol. 62 (1940) pp. 787–791) follows easily. (Received September 11, 1943.)

257. Nelson Dunford and D. S. Miller: *On the ergodic theorem.*

Let $S$ be a space with measure $|e|$ in a $\sigma$-field $F$, and $|S|$ finite. Define $\phi$ to be a 1-1 map of $S$ into all of itself with $\phi^{-1}e \in F$ when $e \in F$. For every real function $f$ on $S$ let $Tf=g$ where $g(t)=f(\phi t)$. In this paper it is shown that a necessary and sufficient condition for the mean ergodic theorem to hold for such an operator $T$ is that $n^{-1}\sum_{r=0}^{n-1}|\phi^{-r}e| \leq M|e|$, $e \in F$, $n=1, 2, \cdots$. It is also proved that in spite of the fact that $\phi$ may not be a measure preserving transformation it is nevertheless still true that the mean ergodic theorem implies the point ergodic theorem. The two above results are also obtained for the $n$-parameter continuous case. (Received August 4, 1943.)

258. Paul Erdös and Hans Fried: *On the connection between gaps in power series and the roots of their partial sums.*

Let $f(z)=1+a_1z+\cdots+a_nz^n+\cdots$ be a power series with the radius of convergence 1. It has Ostrowski gaps if there exists a $\rho<1$ and a pair of infinite sequences $m_k$ and $n_k$ with $\lim_{k\to\infty} m_k/n_k>1$, such that $|a_n|<\rho^n$ if $m_k \leq n \leq n_k$. It has infinite Ostrowski gaps if $\lim_{k\to\infty} m_k/n_k=\infty$. The number of roots of $f_n(z)=1+a_1z+\cdots+a_nz^n$ in the circle of radius $1+r$ is denoted by $A_n^{1+r}$. The following theorems are proved. If $f(z)$ has Ostrowski gaps, then there exists an $r>0$ such that $\liminf_{n\to\infty} A_n^{1+r}/n<1$. This converse is also true. If there exists an $r>0$ such that $\liminf_{n\to\infty} A_n^{1+r}/n<1$, then $f(z)$ has Ostrowski gaps. This theorem is not new. Mr. Bourion (*L'Ultraconvergence dans les Séries de Taylor*, Actualités Scientifiques et Industrielles, no. 472) proved it but his proof is quite different from that given here. Further, the following theorem is proved. If $f(z)$ has infinite Ostrowski gaps, then there exists an $r>0$, such that $\liminf_{n\to\infty} A_n^{1+r}/n=0$. The converse is also true. One of the main tools is the following lemma which may be of interest in itself. Let $F(\rho)$, where $\rho<1$, be the family of all polynomials of the form $1+a_1z+\cdots+a_mz^m+\cdots+a_nz^n$ where $|a_k|<\rho^k$ if $m \leq k \leq n$, and if $r<1/\rho$, then there exists a $c_r$ such that the number of roots of every polynomial belonging to $F(\rho)$ outside the circle with the radius $r$ is greater than $c_r(n-m)$. (Received August 4, 1943.)

259. Herbert Federer: *Surface area.* I.

Suppose $m \leq n$ are positive integers. For each subset $S$ of $n$-space let $\gamma(S)$ be the superior of the $m$-dimensional Lebesgue measures of the perpendicular projections of

$S$ into all $m$-dimensional subspaces of $n$-space. $\gamma_r(S)$ is the inferior of numbers of the form $\sum_{i=1}^{\infty}\gamma(A_i)$ where $A_1, A_2, A_3, \cdots$ are open connected subsets of $n$-space of diameter less than $r$ whose union contains $S$. $\Phi(S)=\lim_{r\to 0+}\gamma_r(S)$. $\Phi$ is an $m$-dimensional measure over $n$-space which agrees with Carathéodory linear measure in case $m=1$ and with Lebesgue measure if $m=n$. The author is interested in the relation of $\Phi$ to $m$-dimensional surfaces in $n$-space. He proves: If $m=2$, $n=3$, $R$ is a rectangle and $f$ is a continuous numerically valued function on $R$, then the Lebesgue surface area of $f$ equals the $\Phi$ measure of the set of all points of the form $(u, v, f(u, v))$ with $(u, v)$ in $R$. (Received September 23, 1943.)

## 260. Herbert Federer: *Surface area.* II.

Retaining the conventions stated in abstract 49-11-259 the author investigates the validity of the formula $\int N(f, T, y)d\Phi y=\int_T Jf(x)dx$ where $f$ is a Lebesgue measurable function on $m$-space to $n$-space and $T$ is a Lebesgue measurable subset of $m$-space. Here $N(f, T, y)$ is the number (possibly $\infty$) of points $x$ in $T$ for which $f(x)=y$; and $Jf(x)=(\det(\overline{L}L))^{1/2}$ where $L$ is the approximate differential of $f$ at $x$ and $\overline{L}$ its conjugate. The formula holds for general $m$ whenever $\limsup_{z\to x}|f(z)-f(x)|/|z-x|<\infty$ for every $x$ in $T$. In the special case $m=2$ (here $Jf=(EG-F^2)^{1/2}$) the formula holds if corresponding to each point $x$ in $T$ there are three distinct unit vectors $\bar{x}^1, \bar{x}^2, \bar{x}^3$ such that $\limsup_{t\to 0+}|f(x+t\bar{x}^j)-f(x)/t|<\infty$ for $j=1, 2, 3$. Finally if $m=n=1$ and if the upper right-hand derivative of $f$ is finite at each point of $T$, then the formula is likewise true and $Jf$ may be replaced in it by the absolute value of the Dini derivative. (Received September 23, 1943.)

## 261. M. H. Heins: *On a problem of Walsh concerning the Hadamard three circles theorem.*

The present paper consists of a contribution to the solution of the following problem proposed to the author by J. L. Walsh: Let $\mathfrak{A}$ consist of the class of functions $f(z)$ which satisfy the following requirements: (a) $f(z)$ is analytic for $z<R(>0)$, (b) $|f(z)|<M(>0)$ for $|z|<R$, (c) $|f(z)|\leq m(<M)$ for $|z|\leq r(<R)$. Under these circumstances it is required to determine l.u.b. $M(f, \rho)$ with $f\in\mathfrak{A}$, where $r<\rho<R$ and $M(f, \rho)=\max_{|z|=\rho}|f(z)|$, and the associated extremal functions. The main concern of the paper is with the descriptive properties of the extremal functions. A method highly suggestive of the typical Tchebycheff argument is used to show that the extremal functions define $(1, k)$ directly conformal maps of $|z|<R$ onto $|w|<M$ and that they are unique when suitably normalized. Under the assumption that $r$ is sufficiently small, the degree $k$ is determined and the associated extremal functions are calculated for the special case where $r^2\leq m/M$. Related questions are treated for the class of functions satisfying (a) and (b) and (c) modified by replacing "$|z|\leq r$" with "$z\in E$" where $E$ belongs to and is closed relative to $-1<x<+1$, and l.u.b. $E<1$. (Received August 6, 1943.)

## 262. M. H. Heins: *On the problem of Milloux for functions analytic throughout the interior of the unit circle.*

Let $E$ denote a point set lying in and closed relative to $|z|<1$ and having the property that it has a non-void intersection with $|z|=r$ for $0\leq r<1$. Let $m$ denote a positive number less than one. Finally let $f(z)$ denote a function analytic and of modu-

lus less than one for $|z|<1$ which satisfies the further condition that there exists an $E$-set $E_f$ such that $|f(z)| \leqq m$ for $z \in E_f$. It is desired to determine l.u.b. $\{\max_{|z|=\rho}|f(z)|\}$ $(0<\rho<1)$ and the corresponding extremal functions. This problem is an extremal problem associated with the original Milloux problem. The extremal value and the corresponding extremal functions are explicitly determined in terms of the Jacobi $sn$-function. Two other methods are given for treating the problem. One employs the theory of Blaschke products, the other gives an algorithm in terms of a process involving the composition of (1, 2) directly conformal maps of the interior of the unit circle onto itself. (Received August 14, 1943.)

263. M. R. Hestenes: *The isoperimetric problem of Bolza in parametric form.*

The purpose of the present paper is to establish a conjecture due to McShane that for a strong relative minimum it is sufficient that for each admissible variation $y$ there is a set of multipliers with which the arc under consideration satisfies the Euler-Lagrange equation, the transversality condition, the Weierstrass condition, non-singularity and such that the second variation is a modification of those used by McShane (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344–379) and Myers (Duke Math. J. vol. 10 (1943) pp. 73–97). An interesting feature of the method is that one obtains the sufficiency theorem for the isoperimetric problem without transformation of the problem to a non-isoperimetric problem. Moreover, one also obtains simultaneously the analogues of the theorems of Osgood. (Received August 4, 1943.)

264. Norman Levinson: *On a nonlinear differential equation of the second order.*

In the equation $\ddot{x}+f(x)\dot{x}+x=e(t)$, $e(t)$ is continuous and periodic of period $L$; $f(x)$ is continuous and greater than 0 except possibly at a finite number of points. If the integral of $f(x)$ over $(x_0, \infty)$ diverges then the above differential equation has a periodic solution of period $L$ and *all* other solutions tend to this solution as $t \to \infty$. (Received September 25, 1943.)

265. Brockway McMillan: *Random point patterns.* Preliminary report.

$X$ is a space, $X^k$ the $k$-fold Cartesian product of $X$ by itself. $W$ is a space whose elements are countable subsets $w=\{x_i\} \subseteq X$. N. Weiner (Amer. J. Math. vol. 60 (1938) pp. 925, 928) exhibited a probability measure $P$ in $W$ such that if $F(w)=f_0+\sum_k \sum' f_k(x_1, \cdots, x_k)$, where the $\sum'$ is over all $k$-tuples of distinct points $x_i \in w$, then (i) $\int F(w)dP=f_0+\sum_k \int f_k(x_1, \cdots, x_k)dm_k=L(F)$, where $m_k$ is a suitable measure in $X^k$. Such $P$'s are characterized by moment conditions on the number of points in $Aw$, $A \subseteq X$. An inverse problem has been treated by N. Wiener and A. Wintner (Abstract 46-11-480): Given $L(F)$ defined by (i), does a probability measure $P$ exist with $L(F)=\int F(w)dP$? The present paper solves this inverse problem by applying previous results about absolutely monotone functions of sets (Abstract 47-11-471). Any completely monotone ($CM$) function of sets $A \subseteq X$ defines a measure in $W$. $P$ exists whenever $1+\sum_k (-1)^k(k!)^{-1}m_k(A^k)$ converges absolutely to a $CM$ function of $A$. (Received August 3, 1943.)

266. A. P. Morse: *A theory of covering and differentiation.*

A covering theory is developed which embraces a variety of theorems of the Vitali type. By substituting a new and frequently useful concept of regularity for the classical notion the author arrives at a general theory of differentiation. This includes and supplements the Lebesgue theory and, at the same time, encompasses results in the differential theory of nets. Use is made of the closed subset theorems alluded to in the abstract which follows. (Received September 11, 1943.)

267. A. P. Morse and J. F. Randolph: *The $\phi$ rectifiable subsets of the plane.*

Suppose $A$ is a plane set, $\phi$ is a measure with $\phi(A) < \infty$, and suppose that, at $\phi$ almost all points of $A$, the upper linear $\phi$ density of $A$ is less than 1.01 times the lower linear $\phi$ density of $A$. Under these circumstances it is shown that all except a $\phi$ small part of $A$ lies on a rectifiable arc. A systematic study of the relations between density and $\phi$ rectifiability is made. In fact by setting $\phi$ equal to Carathéodory linear measure, one of our theorems becomes in essence a compendium of the results obtained by Besicovitch in his paper *On the fundamental geometrical properties of linearly measurable plane sets of points.* II, Math. Ann. vol. 115 (1938) pp. 296–329. Some closed subset theorems are obtained. These involve measure and are apparently heretofore unknown. The paper under discussion together with the paper whose abstract precedes will appear in an early issue of the Transactions. (Received September 11, 1943.)

268. F. J. Murray: *On the existence of quasi-complements in Banach spaces.*

Let $B$ be a reflexive Banach space. Let $M$ and $N$ be closed additive subsets of $B$. $N$ is a quasi-complement of $M$ if $M \cdot N = (0)$ and $M \cdot + \cdot N$ is dense in $B$. It is proved that for every closed additive $M$, such a quasi-complement exists. (Received September 23, 1943.)

269. H. E. Newell: *On the asymptotic forms of the solutions of a linear matric differential equation in the complex domain.*

In an earlier paper (*The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain*, Duke Math. J. vol. 9 (1942)) the author discussed the existence of solutions to the matric differential equation $(d/dx)Y(x, \lambda) = \{\lambda(\delta_{ij}r_j(x)) + (q_{ij}(x, \lambda))\}Y(x, \lambda)$, in which $x$ and $\lambda$ are complex variables. Under certain conditions of suitability imposed upon the coefficient functions and the domains of variation of $x$ and $\lambda$, it was possible to show that regions of existence can be constructed in which the differential equation possesses solutions of the form $P(x, \lambda)E(x, \lambda)$, where $E(x, \lambda) = (\delta_{ij} \exp\{\lambda \int^x r_j(x)dx\})$, and $P(x, \lambda)$, analytic in $x$, reduces uniformly in $x$ to the identity matrix when $\lambda$ becomes infinite. The author applied the elegant and basic concepts of associated and fundamental regions originated by R. E. Langer (*The boundary problem of an ordinary linear differential system in the complex domain*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 151–162, and correction, p. 467) to some of the cases in which the functions $r_j(x)$ have poles and the differences $r_i(x) - r_j(x)$, $i \neq j$, have poles or zeros on the boundary of the $x$ region in question. In adapting the concept of associated regions to the cases considered, the condition was imposed upon the differences $r_i(x) - r_j(x)$, $i \neq j$, that those having simple poles be of the form $a^{ij}(x - x_0)^{-1}$, where $a^{ij}(+0)$ is a constant. The present paper removes that restriction. (Received August 4, 1943.)

270. C. E. Rickart: *An abstract Radon-Nikodym theorem.*

Let $\mathfrak{M}$ be a $\sigma$-field of subsets of an abstract set $M$ and let $m(e)$ be a non-negative measure function defined on $\mathfrak{M}$. This paper contains a proof of the following generalization of the Radon-Nikodym theorem: *If $M$ is the union of a countable number of sets of finite measure and $X(e)$ is any completely additive function defined on $\mathfrak{M}$ to an arbitrary Banach space, then a necessary and sufficient condition that $X(e)$ be representable as an integral, with respect to $m(e)$, is that $X(e)$ be absolutely continuous relative to $m(e)$* (that is, $m(e)=0$ implies $X(e)=0$). The integral used is the Pettis integral (B. J. Pettis, Trans. Amer. Math. Soc. vol. 44) extended (R. S. Phillips, Trans. Amer. Math. Soc. vol. 47) so that it is defined for certain "contractive" functions (that is, multivalued set functions such that $e_1 \subseteq e_2$ implies $f(e_1) \subseteq f(e_2)$). (Received August 6, 1943.)

271. H. M. Schwartz: *Riemann integration with respect to additive interval functions.* Preliminary report.

It is shown first that in order to have finite upper Darboux integrals for continuous integrands, the integrator $g(i)$ (defined and additive over the closed intervals $i$ lying in a fixed finite many-dimensional interval) must be of bounded variation. It is then shown that when $g(i)$ is continuous, all the essential properties of ordinary Riemann integration remain valid. Since every $g(i)$ can be decomposed into a continuous function and a function of jumps (in an extended sense), the essentially new properties are obtained by studying the integration with respect to the latter function. It is also shown that the class of Riemann integrable functions (taken with the Moore-Smith limit) can be generated from the class of simple step functions by employing a natural notion of limit of functions, which is closely related to the one employed by Arzelà and Hobson (cf. E. W. Hobson, *Theory of functions of a real variable*, vol. 2, 3d edition, pp. 312–314). (Received August 7, 1943.)

272. Max Shiffman: *Isoperimetric inequalities and continuity of area for classes of surfaces.*

The isoperimetric inequality $A \leqq L^2/4\pi$ has been extended to minimal surfaces by Carleman, and to sufficiently regular surfaces of nonpositive curvature by Beckenbach and Radó. Advanced methods were used, involving analytic functions or conformal mapping and subharmonic functions. Here it is shown by purely elementary methods that the isoperimetric inequality applies to polyhedra all of whose interior vertices have the sum of face angles greater than or equal to $2\pi$. The result can then be extended to surfaces of nonpositive curvature through approximation by such polyhedra. The isoperimetric inequality can be used as a decisive analytical tool in the estimation of areas of boundary strips. For example, one can establish general theorems involving the continuity of area of surfaces of nonpositive curvature. See, for example, Shiffman, *Unstable minimal surfaces with any rectifiable boundary*, Proc. Nat. Acad. Sci U.S.A. vol. 28 (1942) pp. 103–108, where minimal surfaces are considered. (Received October 1, 1943.)

273. Dorothy M. Stone: *The representation of abstract measure functions.*

In this paper a standard form is obtained for general abstract measure algebras in which the measures have values in abelian semi-groups. Incidentally it is shown

that the theory of such abstract measures can be reduced to the theory of ordinary numerical measures. Let $E$ be a non-atomic Boolean $\sigma$-algebra satisfying the countable chain condition. Let $\lambda$ be an abstract measure-function on $E$; $\lambda$ is assumed to be countably additive and non-atomic. Writing $a \sim b$ if $\lambda(a)=\lambda(b)$, an equivalence relation between certain pairs of elements of $E$ is defined, which has four simple properties. Conversely, any equivalence relation with these properties determines such a measure $\lambda$, uniquely to within isomorphism. Working only with the equivalence relation it is shown that there exists a finite isomorphism $\phi$ of $E$ onto a certain subalgebra of a direct sum of measure algebras $E_\alpha$, each having a numerical-valued measure $\mu_\alpha$, in such a way that $a \sim b$ if and only if $\mu_\alpha[\phi_\alpha(a)]=\mu_\alpha[\phi_\alpha(b)]$ (for all $\alpha$), where $\phi_\alpha(x)$ denotes the projection of $\phi(x)$ onto $E_\alpha$. A $\sigma$-representation can be deduced from this by reducing modulo a certain $\sigma$-ideal in the direct sum. (Received October 2, 1943.)

## 274. J. V. Wehausen: *A remark concerning a set of completely continuous transformations.*

Let $E$ be a Banach space, $K$ a closed convex subset of $E$, $\mathfrak{T}$ the set of all completely continuous transformations defined in $E$ with $T(K) \subset K$, $\mathfrak{T}'$ the subset of $\mathfrak{T}$ consisting of transformations with unique fixed points in $K$, and $\mathfrak{T}''$ the subset consisting of those with more than one fixed point in $K$. From the fixed point theorem of Schauder $\mathfrak{T}=\mathfrak{T}'+\mathfrak{T}''$. Let $\mathfrak{T}$ be metrized by $\|T_1-T_2\|=\sup_K \|T_1x-T_2x\|$. Then $\mathfrak{T}$ is complete and the following theorem holds. If $\mathfrak{T}'$ is dense in $\mathfrak{T}$, then $\mathfrak{T}'$ is of the second category and $\mathfrak{T}''$ of the first category of $\mathfrak{T}$. This generalizes a result of Orlicz (Bulletin international de l'Académie des sciences de Cracovie A, Nr. 8/9 (1932) pp. 221–228) concerning the differential equation $y'=f(x, y)$. (Received September 24, 1943.)

## 275. J. E. Wilkins: *Multiple integral problems in parametric form in the calculus of variations.*

This paper discusses the problem of minimizing a multiple integral in a class of varieties defined parametrically. In order that the multiple integral be independent of the parametric representation, it is necessary and sufficient that the integrand function satisfy a certain homogeneity condition. The implications of this homogeneity condition for the derivatives of the integrand of arbitrary order are investigated. The usual necessary conditions known in the non-parametric case for multiple integrals are extended to the parametric case. A discussion of the form of the Weierstrass condition is given. Finally, as a first step in obtaining a field theory, necessary and sufficient conditions that an integral be invariant are given. (Received August 5, 1943.)

## 276. Antoni Zygmund: *On certain integrals.*

(a) Given a function $\phi(z)$ regular for $|z|<1$ and of the class $H^r$, $r>1$, let $g^*(\theta)=\{\int_0^1(1-\rho)\chi^2(\rho, t)d\rho\}^{1/2}$, where $\chi^2(\rho, \theta)=\pi^{-1}\int_0^{2\pi}|\phi'(\rho e^{i(\theta+t)})|^2 P(\rho, t)dt$, $P(\rho, t)$ denoting the Poisson kernel. Then the ratio $M_r(g^*(\theta))/M_r(\phi(e^{i\theta}))$ is contained between two positive numbers depending on $r$ only. (b) Let $f(\theta)$ be of period $2\pi$ and of the class $L^r$, $r>1$. Let $F(\theta)$ be the indefinite integral of $f$, and $\mu(\theta)=\{\int_0^\pi t^{-2}[F(\theta+t)+F(\theta-t)-2F(\theta)]dt\}^{1/2}$. The ratio $M_r(\mu)/M_r(f)$ is again comprised between two positive numbers depending on $r$ only. (c) Every lacunary series $\sum c_k$ which is absolutely summable $A$ must be absolutely convergent. (Received September 10, 1943).

## APPLIED MATHEMATICS

277. Garrett Birkhoff: *The reversibility paradox and camber.*

A comparison is made between the predictions of the (reversible) Kutta-Joukowsky lift theory and the experimental findings of Eiffel, for circular airfoils. It is found that the predicted angle of the chord to the airfoil with the airstream in the position of zero lift ("first axis") varies from two to seven times the observed angle. Whereas with camber 1/7, the observed increase in lift/effective angle of attack is about 83 per cent; the predicted increase 4 per cent. A possible irreversible explanation of this is pointed out. (Received August 14, 1943.)

278. H. B. Curry: *The method of steepest descent for nonlinear minimization problems.*

A practical method for calculating approximately a stationary value of a function $G(x_1, x_2, \cdots, x_n)$ is desirable in connection with certain nonlinear least square problems (abstract 49-11-286). Such a method may be exhibited as follows. Let $x_i^0$ be an initial approximation. The direction in which $g(t) = G(x_i^0 + t\xi_i)$ decreases most rapidly is obtained by putting $\xi_i = -\partial G/\partial x_i$. By trial $t_0$ can be determined so that $g(t_0) < g(0)$. Then the point $x_i^0 + t_0\xi_i$ can be used as a new approximation and the process repeated. It is shown here that if $t_0$ is chosen so that $g'(t_0) = 0$, $g'(t) < 0$ for $0 \leqq t < t_0$, then the successive approximations converge to a point where $\partial G/\partial x_i = 0$ provided that $G$ takes a smaller value at $x_i^0$ than at any point on the boundary of that region $S$ within which differentiability and other usual conditions hold. Although the process is well known in analysis, it does not appear to have been noticed recently in this connection. It is applicable when the initial approximation $x_i^0$ is too rough for the standard least square procedure. The problem includes that of solving $n$ simultaneous nonlinear equations in $n$ unknowns which was handled by Cauchy. (Received October 1, 1943.)

279. D. W. Dudley and Hillel Poritsky: *The geometry of cutting and hobbing of worms and gears.*

The exact shape of the teeth of a worm $W$ or a helical gear $G$ which is produced by a milling cutter $C$ or a hob $H$ of a given tooth profile is investigated. This problem and its converse are of great interest in the manufacture of gears and worms. Assuming numerous cutter teeth one may replace the cutter $C$ by a surface of revolution $S$. In its motion relative to $W$, $S$ occupies a one-parameter family of positions whose envelope is $W$. The curve of contact of $S_0$, any position of $S$, with the envelope represents $C$, the curve of deepest cutting. $C$ may be determined from the condition that along it the velocity of the motion of $S$ relative to $W$ is tangent to $S$. In the converse problem where $W$ is given and $S$ is sought, the motion of $W$ relative to $S$ is utilized in a similar way; this motion can now be simplified to a rotation about the cutter axis. A similar treatment applies to the hob-gear problem except that here a two-parameter family of motions is involved and two kinematic conditions are now required to determine the deepest cutting. (Received August 12, 1943.)

280. Bernard Friedman: *A method of approximating the complex roots of polynomial equations.*

By using successive divisions, an iterative process is set up to approximate the quadratic factors of a polynomial. In this way, the complex roots of largest and small-

est absolute value can be determined. Conditions for convergence and for the rapidity of convergence are also investigated. (Received August 31, 1943.)

281. R. E. Gaskell: *On moment balancing in structural dynamics.*

The moment balancing process (H. Cross, Transactions of the American Society of Civil Engineers vol. 96 p. 1) is developed for the dynamics of structures with the help of the dynamic form of the four-moment equation (W. Prager, Ingenieur-Archiv, vol. 1, p. 527). This combination gives a method of finding stresses in plane, rigid-frame structures, each of whose bars is of uniform cross section and whose joints are not subject to translation, when an oscillating load is applied. The method converges at least for oscillations whose frequency is less than the lowest natural frequency of the structure. (Received August 5, 1943.)

282. G. H. Handelman: *On a principle of M. A. Sadowsky.* Preliminary report.

Sadowsky (Journal of Applied Mechanics vol. 10 (1943) pp. A-65) has applied a heuristic principle of maximum plastic resistance to several types of combined states of stress. The principle has been proved by Prager for a beam under combined tension and torsion. This paper is concerned with the additional case of a symmetric beam subject to bending and torsion. The basic differential equation, derived from the stress-strain relations and the yield condition, is shown to be the Euler-Lagrange equation for the following variational problem: to find that stress distribution for a given applied torque for which the value of the bending couple is stationary. (Received August 9, 1943.)

283. Carl Holtom: *Permanent configurations in the n-body problem.*

Let $n$ masses in a plane be projected with initial velocities and thereafter be allowed to move only under the force of attraction according to the Newtonian law. If the potential function of the system is $U(x_i, y_i)$, then the masses $m_i$ at points $(x_i, y_i)$ form a permanent configuration if there exists a real solution of the equations $\partial U/\partial x_i + m_i\omega^2 x_i = 0$ and $\partial U/\partial y_i + m_i\omega^2 y_i = 0$, where $\omega^2$ is the angular velocity of the system. It is shown that for the case of $n-1$ masses there exists at least one real point of liberation at which $m_n = 0$ may be placed, and that the solutions vary continuously and remain real as $m_n$ increases to a positive value. For the case $n > 4$ a restriction on the masses or mass ratios is sufficient, though it may not be necessary, to insure the existence of a permanent configuration. The equations may be solved for the coordinates as functions of the mass ratios, and several terms in the series solutions are given. (Received August 6, 1943.)

284. L. C. Hutchinson: *Free vibrations in a rectangular rod.* Preliminary report.

In this paper the author studies, directly from the general equations of motion, the possible free vibrations in a rectangular rod of isotropic homogeneous material, and their relationship to the corresponding acoustic and electromagnetic waves. (Received August 6, 1943.)

285. Arthur Korn: *Compressibility with repect to vibrations of high frequency.*

When the velocities and accelerations remain under a certain limit, the compressibility of the air can be quite well handled by means of the well known state equation $p=R\rho T$, in which $p$ is the pressure, $\rho$ the density, $T$ the absolute temperature and $R$ a constant. If one takes account of friction the formula is slightly modified. For high values of velocities and accelerations, especially when one has to do with vibrations of high frequency, new approximations have to be found. The state equation of compressible matter not only contains according to the author's ideas all the secrets of quantum mechanics, but also has important relations to the theory of the electromagnetic field. For vibrations $\mathfrak{v}=\mathfrak{v}_0+\mathfrak{v}_1\cos\nu t+\mathfrak{v}_2\sin\nu t$ with frequencies, as they are considered in quantum mechanics, for example, in the theory of the spectra, the state equation may have the form $p=R\rho T+R'\nu\rho(T)^{1/2}$, where $R$ and $R'$ are constants and $\nu$ the frequency of the vibration. For high frequencies the pressure will become more and more proportional to the frequency. When the frequency still becomes higher and higher, one may make the assumption that for such vibrations approximate incompressibility takes place. (Received August 12, 1943.)

286. Kenneth Levenberg: *A method for the solution of certain non-linear problems in least squares.*

The standard method of least squares for problems involving a set of residual functions $f_i(\alpha, \beta, \gamma, \cdots)$, $i=1, 2, \cdots, n$, which are nonlinear in the parameters $\alpha, \beta, \gamma, \cdots$, consists of reducing the $f_i(\alpha, \beta, \gamma, \cdots)$ to linear form by first order Taylor approximations $F_i(\alpha, \beta, \gamma, \cdots)$, taken about an initial trial solution for the parameters $\alpha_0, \beta_0, \gamma_0, \cdots$, and minimizing $S=\sum_1^n F_i^2(\alpha, \beta, \gamma, \cdots)$ by solving the corresponding linear normal equations (Whittaker and Robinson, Calculus of Observations, 1937, p. 214). If $\alpha_1, \beta_1, \gamma_1, \cdots$ are the solution of these normal equations, it may happen, especially in certain engineering applications, that $\sum_1^n f_i^2(\alpha_1, \beta_1, \gamma_1, \cdots) \geqq \sum_1^n f_i^2(\alpha_0, \beta_0, \gamma_0, \cdots)$ in which case the process has failed to improve the initial solution. This paper offers a method of solving the problem in this case. It is shown that the initial solution can be improved by the minimization of $wS+Q$, where $Q$ is a positive definite quadratic form in $\alpha-\alpha_0, \beta-\beta_0, \gamma-\gamma_0, \cdots$, and $w$ is a sufficiently small positive quantity for which an approximate expression, which may be improved by trial, is given. For $Q=(\alpha-\alpha_0)^2+(\beta-\beta_0)^2+(\gamma-\gamma_0)^2+\cdots$, the method consists of revising the normal equations by the addition of a constant $1/w$ to the coefficients of the principal diagonal. (Received October 1, 1943.)

287. L. L. Merrill: *The mathematical determination of cooling rates during arc welding.*

This is an extension of work done by Dr. Daniel Rosenthal who determined the temperature distribution in very thick or very thin plates during arc welding under the simplifying assumptions that the heat source is a point source and the plate metal in the neighborhood of the source undergoes no change. The present investigation considers plates of various finite thickness and, by introducing an "input factor," takes into account the facts that the heat source is not a point source and that a pool of molten metal trails the moving electrode. To accomplish this two temperature-time records for each plate thickness must be determined experimentally at a point at the edge of the molten metal, one with the plate initially at room temperature and

the other with the plate at some convenient pre-heat temperature. It is then possible to predict the cooling rate at any temperature during the cooling, for any initial plate temperature, for any travel speed, and for any heat input within the range of interest. Results of the investigation make it possible to determine welding conditions necessary to produce any desired cooling rate at any temperature level during cooling and thus to make a weld with desired characteristics. (Received August 4, 1943.)

288. A. D. Michal: *An analogue of the Maupertuis-Jacobi "least" action principle for dynamical systems of infinite degrees of freedom.*

More than a century ago, Jacobi gave his famous least action principle for the dynamical trajectories of dynamical systems with finite degrees of freedom. Although Hamilton's principle has been used widely in the derivation of the equations of motion for infinite degree of freedom problems in physics and engineering, analogues of the Jacobi least action principle do not seem to be in the literature. In this paper, analogues of the Jacobi least action principle are given for various infinite degree of freedom problems. The functionals that are made stationary are more general than those considered in the usual calculus of variations. (Received August 6, 1943.)

289. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* I. *Wave motion as a study in geodesics.*

In this paper, the second order partial differential equations of wave motion are shown to define dynamical states that can be considered as generating geodesics in some of the author's general "Riemannian" spaces with linear topological coordinates (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 529–563, especially pp. 551–559). The author's earlier geometries (1927–1931) in infinitely dimensional spaces are not suitable for these particular physical models partly because some of the functionals have a higher order continuity than zero. (Received August 6, 1943.)

290. Isaac Opatowski: *Cantiliever beam of circular cross section and constant strength under the action of its own weight.*

The determination of the cross section consists in the solution of a nonlinear integral equation, which gives the profile of the beam in the shape of two parabolas tangent at their common vertex. A calculation of deflections, based on the usual approximation concerning the curvature, gives an infinite deflection at the free end. The use of an exact expression of the curvature gives an imaginary deflection curve, at least within a certain range containing the free end. These impossible conclusions are due to the use of the theory of prismatic beams for a beam of variable cross section. An extension is made to the case in which, besides the beam's own weight, a known force $F$ acts at its free end. This leads to a hyperelliptic integral which gives a known formula when the beam's own weight is negligible with respect to $F$ (see, for example, Hütte, *Des Ingenieurs Taschenbuch*, 25th edition, Berlin, 1925, vol. 1, p. 627). (Received August 6, 1943.)

291. M. O. Peach: *Simplified technique for constructing orthonormal functions.*

To orthonormalize the functions $f_1, f_2, \cdots$ over a given region $R$ evaluate the quantities $d_{p,q}=\int_R f_p \bar{f}_q dR$, $0<p, q\leq n$, where $n$ orthonormal functions are desired.

A computer, whose mathematical attainments need extend no further than a knowledge of the algebraic rules of sign, may complete the calculations. The quantities $d_{p,q}$ are formed into a matrix, the unit matrix is adjoined, and a process $P$ is applied to the combined matrix which reduces its order by one. Repeated application of $P$ yields directly the coefficients of all $n$ orthogonal functions. A root extraction is required to obtain the normalizing factor for each function. The process $P$ consists of the repeated evaluation of certain second-order determinants. Proof depends upon certain properties of $P$ and is obtained largely from the elementary properties of determinants. The method is particularly adapted to the use of modern computing machines. (Received August 9, 1943.)

292. A. E. Ross: *A note on forced oscillations of a conducting sphere.*

J. A. Stratton and L. J. Chu (Journal of Applied Physics vol. 12 (1941) pp. 236–240) studied the problem of forced oscillations in a conducting sphere. Taking the equatorial line as the effective region of an applied oscillatory field $E'$, they obtained a series expansion of the current $I_\theta$ crossing the equatorial plane and used it to compute and plot the curves for the radiation conductance, and susceptance, and input admittance of the first few modes of the sphere, and also for the total radiation conductance. It was noted by Stratton and Chu that the above series expansions were not convergent. In the present paper, at the suggestion of Professor Brillouin, the author takes an equatorial strip of finite fixed width $\Delta\theta$ as the effective region of the applied field $E'$, shows that the resulting expansion for the current $I_\theta$ through the surface $\theta=$const. is convergent, and employs it to plot the graphs of the above mentioned quantities. It is of interest to observe that to different values of $\Delta\theta$ correspond different selections of dominant modes in the expansions of the input impedance $Y_\theta$. (Received August 6, 1943.)

293. J. A. Shohat: *Parseval formula in its application to Van der Pol's and generalized equations.*

First, an error is indicated in the paper by the author: *A new analytical method* · · ·, Journal of Applied Physics vol. 14 (1943) pp. 40–48. It is further shown, by a simple application of the Parseval formula for Fourier series, that the main result of Van der Pol—approximating the solution by a fundamental oscillation—is mathematically correct, and the error of such an approximation is estimated. Its amplitude is derived anew, with the same numerical result as previously obtained. The present method is shown to apply to the more general equation $d^2u/dt^2-\epsilon F(u)du/dt+u=0$. (Received September 15, 1943.)

294. C. A. Truesdell: *The differential equations of the membrane theory of shells of revolution.*

Previously the "membrane equations" have been derived from a figure. Here they are shown to be consequences of the general equations of elasticity. First a generalization of spherical polar coordinates is introduced in which one family of coordinate surfaces is that obtained by revolving curves parallel to some fixed curve about an axis. Then the assumption is made that there is no force normal to the shell across any cut, and no force component in any direction across the surface of the shell or across any parallel surface. There result equations similar to those usually given for the membrane theory and not more difficult to solve. If the thickness of the shell is small compared with its radii of curvature, the usual equations can be deduced, in

which the stress does not vary along a normal. The significance of the assumptions is discussed. Finally it is shown that solutions of the equations of the membrane theory lead to a state of strain which does not satisfy the conditions of compatibility, so that displacements calculated in the membrane theory violate geometry. (Received August 6, 1943.)

295. D. V. Widder: *Positive temperatures on an infinite rod.*

In this paper it is proved that any solution $u(x, t)$ of the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial t$ which is non-negative for positive $t$ and which vanishes for $t=0$ is identically zero. By use of this result it is shown that any solution which is non-negative for $t>0$ has the Poisson-Stieltjes representation $u(x, t)=\int_{-\infty}^{\infty} k(x-y, t)d\alpha(y)$. Here $\alpha(y)$ is nondecreasing and $k(x, t)$ is the familiar source solution $(4\pi t)^{-1/2}\exp(-x^2/4t)$. As a consequence any such solution is analytic in $x$ and in $t$. (Received August 24, 1943.)

## GEOMETRY

296. R. C. Buck: *Partition of space.*

By an application of elementary topology, it is shown that $n$ hyperplanes, with general intersection, partition Euclidean $r$-space into $M_r(p, n)$ $p$-dimensional regions, where $M_r(p, n)=\sum_{r-p}^{r} C_{n,k}C_{k,r-p}$, of which $C_{r,p}C_{n,r+1}$ $(r+1)/(n+p-r)$ are bounded. The problem is also solved for projective $r$-space, yielding $\sum_{0}^{[(r-p)/2]} C_{r-2k,r-p}C_{n,r-2k}$ as the number of $p$-dimensional regions. This completely solves the well known "cheese slicing" problem. (Received September 11, 1943.)

297. John DeCicco: *Dynamical and curvature trajectories in space.*

Kasner has studied the geometry of dynamical trajectories in the plane and in space in the Princeton Colloquium (Amer. Math. Soc. Colloquium Publications, vol. 3). This paper considers the problem of determining all quintuply-infinite systems of curves in space which are at once dynamical and curvature trajectories. In the plane, Kasner has shown that the appropriate families are the trajectories of all central or parallel fields of force. It is shown that the systems of $\infty^5$ curves which are simultaneously dynamical and curvature trajectories are the dynamical trajectories of the following three distinct types of fields of force: (I) Those whose lines of force all lie in a pencil of planes. (II) Those whose lines of force are orthogonal to a family of $\infty^3$ circular helices, all of which possess the same axis and the same period. (III) Those of the central or parallel type. Each of these types is projectively invariant. (Received August 11, 1943.)

298. Edward Kasner and John DeCicco: *Union-preserving transformation of space.*

Sophus Lie showed that the only lineal-element transformations of the contact type are the extended point transformations. This result is extended by studying transformations from differential curve-elements of order $n$: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where $n$ is 2 or more, into lineal-elements $(X, Y, Z, Y', Z')$. The entire class of the union-preserving transformations is determined. Any general union-preserving transformation from curve-elements of order $n$ into lineal-elements is completely determined by a new directrix equation $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)})=0$. The only available union-preserving transformations (in the whole domain of

curve-elements) are the group of point transformations and the union-preserving transformations from curve-elements of order $n$ into lineal-elements, or the extensions of these two types. An additional theorem is that any transformation from curve-elements of the $(x, y, z)$-space into lineal-elements of the $(X, Y, Z)$-space, by which any union of the $(X, Y, Z)$-space corresponds to exactly $\infty^{2(n-1)}$ curves of the $(x, y, z)$-space, is union-preserving. (Received August 11, 1943).

299. L. A. MacColl: *Geometrical characterizations of some families of dynamical trajectories.*

This paper deals with a certain five-parameter family of curves, which can be regarded as the family of trajectories of an electrified particle in an arbitrary static magnetic field. A set of geometrical properties is given which completely characterizes the family of curves. Certain other related families of curves, including the four-parameter family of trajectories of the particle moving with an arbitrarily prescribed value of the energy, are also discussed and characterized by sets of geometrical properties. (Received October 1, 1943.)

300. Alice T. Schafer: *Two singularities of space curves.*

This paper uses the methods of projective differential geometry to study an analytic space curve in the neighborhood of an inflexion point and, second, a planar point. Canonical power-series expansions representing the curve in the neighborhood of each singular point are deduced by suitably choosing the projective coordinate system. These canonical expansions are then used to study properties of the curve in this neighborhood. Particular emphasis is placed on the surfaces osculating the curve, sections of the tangent developable of the curve made by the faces of the tetrahedron of reference, and projections of the curve onto the faces of the tetrahedron of reference. (Received October 1, 1943.)

## Statistics and Probability

301. W. K. Feller: *On a general class of "contagious" distributions.*

This paper is concerned with some properties of a class of contagious distributions which contains, among others, some distributions studied by Greenwood and Yule, Polya, and Neyman, respectively. (Received August 3, 1943.)

302. H. B. Mann and Abraham Wald: *On the statistical treatment of linear stochastic difference equations.*

For any integer $t$ let $x_{1t}, \cdots, x_{rt}$ be a set of $r$ random variables which satisfy the system of linear stochastic difference equations $\sum_{j=1}^{r}\sum_{k=0}^{p_{ij}}\alpha_{ijk}x_{j,t-k}+\alpha_i=\epsilon_{it}(i=1, \cdots, r)$. The coefficients $\alpha_{ijk}$ and $\alpha_i$ are (known or unknown) constants and the vectors $\epsilon_t=(\epsilon_{1t}, \cdots, \epsilon_{rt})$ $(t=1, 2, \cdots)$ are independently distributed random vectors each having the same distribution. It is assumed that $E(\epsilon_{it})=0$. The problem dealt with in this paper is to estimate the unknown coefficients $\alpha_{ijk}$ and $\alpha_i$ on the basis of $Nr$ observations $x_{it}(i=1, \cdots, r; t=1, \cdots, N)$. The statistics used as estimates of the unknown coefficients are identical with the maximum likelihood estimates if $\epsilon_t$ is normally distributed. The joint limiting distribution of these estimates is obtained without assuming normality of the distribution of $\epsilon_t$. (Received August 7, 1943.)

303. Abraham Wald and Jacob Wolfowitz: *An exact test for randomness in the non-parametric case based on serial correlation.*

Let $X_1, \cdots, X_n$ be $n$ chance variables, about the distribution of which nothing is known. Let the problem be to test the (null) hypothesis that $X_1, \cdots, X_n$ are independently distributed with the same distribution function. It is shown that an exact test of this hypothesis based on the serial correlation coefficient can be made. For this purpose the distribution of the serial correlation coefficient in the sub-population consisting of all possible permutations of the observed values is employed. Under the null hypothesis this distribution is independent of the distribution function of $X_i (i=1, \cdots, n)$. Several exact moments are obtained and asymptotic normality is proved. (Received August 7, 1943.)

## TOPOLOGY

304. L. M. Blumenthal: *Metric study of generalized elliptic spaces.* Preliminary report.

Let $\Sigma$ be semimetric with diameter $d$, $\phi(x/\rho)$ a real single-valued monotonic decreasing function ($\rho$, positive parameter) defined over the distance set of $\Sigma$, with $\phi(0)=1$, $\phi(d/\rho)=0$. The space is called generalized elliptic $E^{\phi}_{n,\rho}$ provided: I. For each positive integer $k$ and each $k+1$ points $p_1, p_2, \cdots, p_{k+1}$ there corresponds an *allowable matrix* $(\epsilon_{ij})$; $\epsilon_{ii}=1$, $\epsilon_{ij}=\epsilon_{ji}=\pm 1$ $(i,j=1,2,\cdots,k+1)$ with every nonvanishing principal minor of the determinant $|\epsilon_{ij}\phi(p_ip_j/\rho)|$ positive. II. The integer $n$ is the smallest for which there exist $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ such that $|\epsilon_{ij}\phi(p_ip_j/\rho)|$ does not vanish for any allowable matrix $(\epsilon_{ij})$. For the ordinary elliptic space, $\Sigma$ is the surface of the sphere $S_{n,\rho}$ with opposite points identified and "shorter arc" metric, while $\phi(x/\rho)\equiv\cos(x/\rho)$. An interesting feature of these spaces is that, in contrast to others (that is, euclidean, hyperbolic, spherical) the mutual distances of a set of points does not suffice to determine the dimension of the subspace which contains them. Thus a given set of three numbers may be distances of three points on an $E^{\phi}_{1,\rho}$, and also distances of three points not on any $E^{\phi}_{1,\rho}$, but on an $E_{2,\rho}$. It is found that certain pseudo-$E^{\phi}_{n,\rho}(n+3)$-tuples are contained in an $E^{\phi}_{n+2,\rho}$. New theorems concerning determinants are a by-product of the study. (Received August 3, 1943.)

305. L. M. Blumenthal: *New formulations of some imbedding theorems.*

The theorems deal with congruent imbedding of metric spaces in Hilbert space, and center about the two following results: I. A complete connected ptolemaic metric space in which every point is contained in a closed convex neighborhood is convex. II. A complete, convex, externally convex metric space in which the Theorem of Pythagoras is valid is congruently contained in Hilbert space. An application of a well known theorem of Menger-Schoenberg yields the first result when it is shown that each two points of the space are joined by an arc with everywhere vanishing metric curvature. To establish the second theorem one notes that every pair of lines that intersect at "right angles" is congruently imbeddable in the plane. It follows that the space has the *weak euclidean four-point property* and the conclusion follows from a result due to the writer. (Received August 3, 1943.)

306. R. H. Fox: *The complete homotopy group.*

The fundamental group $\tau_\omega(Y)=\pi_1(Y^T)$ of the space of continuous mappings of the infinite dimensional torus $T$ into a topological space $Y$ has the homotopy groups $\pi_n(Y)$, $n=1, 2, \cdots$, as subgroups. The infinite symmetric group $\{P\}$ is a group of operators on $\tau_\omega$. The complete homotopy group $\tau$ is the minimal subgroup of $\tau_\omega$ containing all the subgroups $\pi_n^P$. The group $\tau$ can be given a direct geometric definition and can be built up from the homotopy groups by a countable number of splitting extensions. The classical automorphisms of $\pi_n$ induced by elements of $\pi_1$ are induced by inner automorphisms of $\tau$. In fact the Abe group $\kappa_n$, $n\geqq 2$, (Jap. J. Math. vol. 16 (1940) p. 169) is the minimal subgroup containing $\pi_1$ and $\pi_n$. Similarly the multiplication $\alpha\cdot\beta\in\pi_{m+n-1}$, $\alpha\in\pi_m$, $\beta\in\pi_n$ (Whitehead, Ann. of Math. vol. 42 (1941) p. 411) is represented as a commutator $\alpha\beta^P\alpha^{-1}\beta^{-P}$ where $P$ is a certain type of permutation. The commutators $\alpha^Q\beta^R\alpha^{-Q}\beta^{-R}$ which are not $\{P\}$-automorphs of commutators of this type are all $=1$. Generalization of the theory to groups modulo a subset $B$ of $Y$ is automatic. (Received October 1, 1943.)

307. D. W. Hall: *A note on primitive skew curves.*

It is shown that a locally connected continuum $M$ not separated by the removal of any pair of its points can contain no primitive skew curve of type II unless it contains a primitive skew curve of type I. As a corollary there is the theorem of F. B. Jones (Abstract 48-11-340) to the effect that a locally connected continuum $M$ separated by no pair of its points but by every one of its simple closed curves must be homeomorphic with a sphere provided $M$ contains no primitive skew curve of type I. This follows immediately in view of a theorem of S. Claytor (Ann. of Math. vol. 35 (1934) pp. 809–835). (Received August 3, 1943.)

308. M. E. Shanks: *The space of metrics on a compact metrizable space.* II.

Denote by $M_0(X)$ the set of all metrics on the compact metrizable space $X$ compatible with its topology, and by $M(X)$ the complete extension of $M_0(X)$. Then $M_0(X)$ and $M(X)$ are semi-linear normed spaces. In a previous abstract 47-5-284 the author stated that $X$ and $Y$ are homeomorphic if and only if $M(X)$ and $M(Y)$ are congruent. In this paper the same result is obtained from the stronger statement that the homeomorphism of $X$ and $Y$ implies, and is implied by, the linear isomorphism $M_0(X)$ and $M_0(Y)$. The methods are new and make essential use of the lattice of upper semi-continuous decompositions of $X$. (Received October 1, 1943.)

309. J. W. T. Youngs: *On surfaces of class $K_1$.* Preliminary report.

In a paper entitled *On the semi-continuity of double integrals in parametric form* (Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361) Radó has proved a variety of important theorems concerning a class of surfaces designated by the symbol $K_1$. The purpose of this paper is to show that every surface is in the class $K_1$. A new functional $I^*(S)=\sup I(T, B, \|X\|)$ is discussed, the supremum being taken with respect to all representations $(T, B)$ of the surface $S$ for which $I(T, B, \|X\|)$ has meaning. (For the terminology see the paper of Radó.) (Received September 22, 1943.)

## NEW PUBLICATIONS

BALLOU, D. H., and STEEN, F. H. Plane and spherical trigonometry with tables. Boston, Ginn, 1943. 6+179+84 pp. $2.40.

HUTCHINSON, C. A. See Rider, P. R.

KELLOGG, O. D. Foundations of potential theory. New York, Murray Printing Co., 1943. Photographic reproduction of 1929 edition. 9+384 pp. $6.00.

KELLS, L. M. Calculus. New York, Prentice-Hall, 1943. 8+509 pp. $3.75.

LEIGHTON, H. L. C. Solid geometry and spherical trigonometry. New York, Van Nostrand, 1943. 19+210 pp. $2.20.

DE LOSADA Y PUGA, C. Galileo. Lima, Universidad Catolica del Peru, 1942. 54 pp.

MURSI, Z. Tables of Legendre associated functions. Cairo, Egypt, Fouad I University. 8+283 pp.

PICCARD, S. Sur des ensembles parfaits. Sur les ensembles de sommes. (Mémoires de l'Université de Neuchatel, vol. 16.) Paris, Librairie Gauthier-Villars, 1942. 196 pp. Fr.s. 7.50.

RIDER, P. R., and HUTCHINSON, C. A. Navigational trigonometry. New York, Macmillan, 1943. 9+232 pp. $2.00.

TRELOAR, A. E. Correlation analysis. Minneapolis, Burgess, 1942. 64 pp. $1.50.

TRELOAR, A. E. Random sampling distributions. Minneapolis, Burgess, 1942. 94 pp. $2.25.

WEYL, F. J. See WEYL, H.

WEYL, H., and WEYL, F. J. Meromorphic functions and analytic curves. (Annals of Mathematics Studies, no. 12.) Princeton University Press; London, Humphrey Milford and Oxford University Press, 1943. 9+269 pp. $3.50.